Superradiant instabilities of AdS black holes

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Based on: CQG **33**, 125022 (2016); 1512.02644 [gr-qc]
PRL **116**, 141102 (2016); 1601.01384 [gr-qc]

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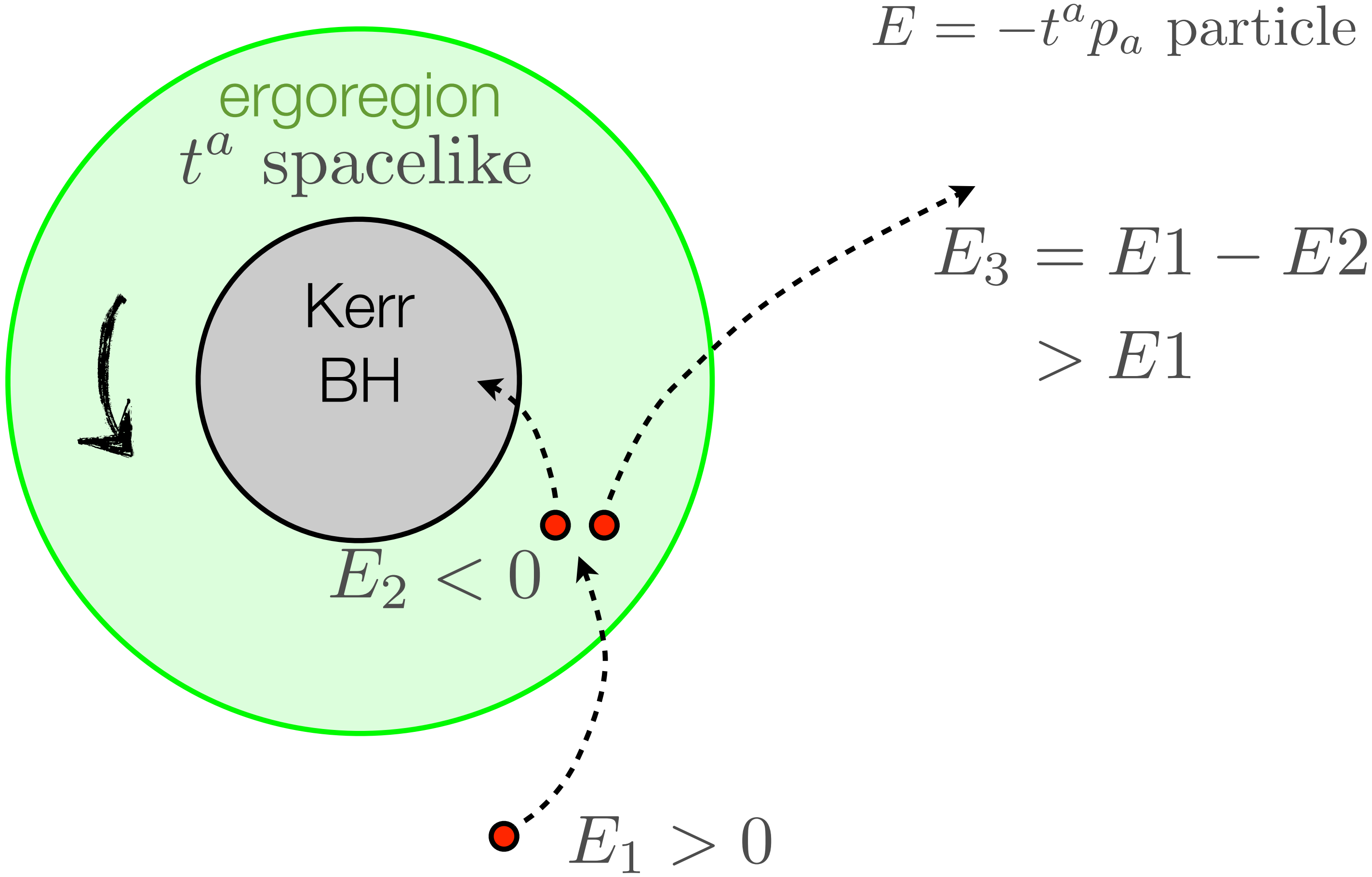
Introduction to superradiant instability

- Mass and angular momentum can be extracted from a black hole with ergoregion.

  E.g., Penrose process

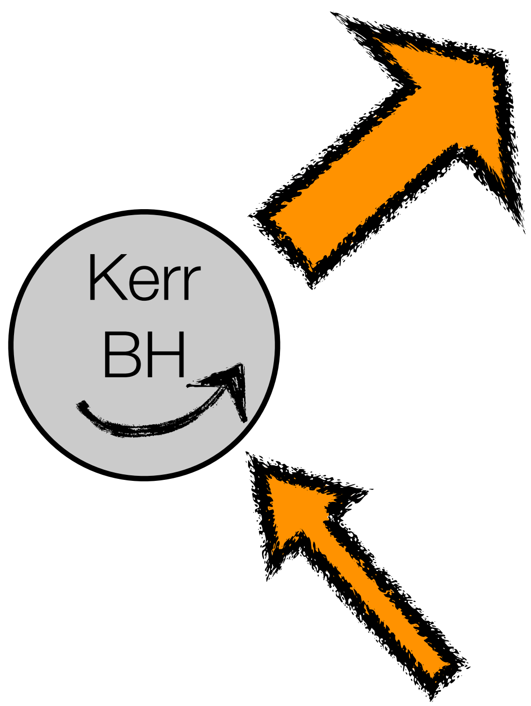
  t^a time translation Killing field
  $E = -t^a p_a$ particle energy



- Area law not violated since $A = 8\pi M\left[M + (M^2 - a^2)^{1/2}\right]$ and particles extract angular momentum as well.

Introduction to superradiant instability

- Similar process amplifies waves: superradiance

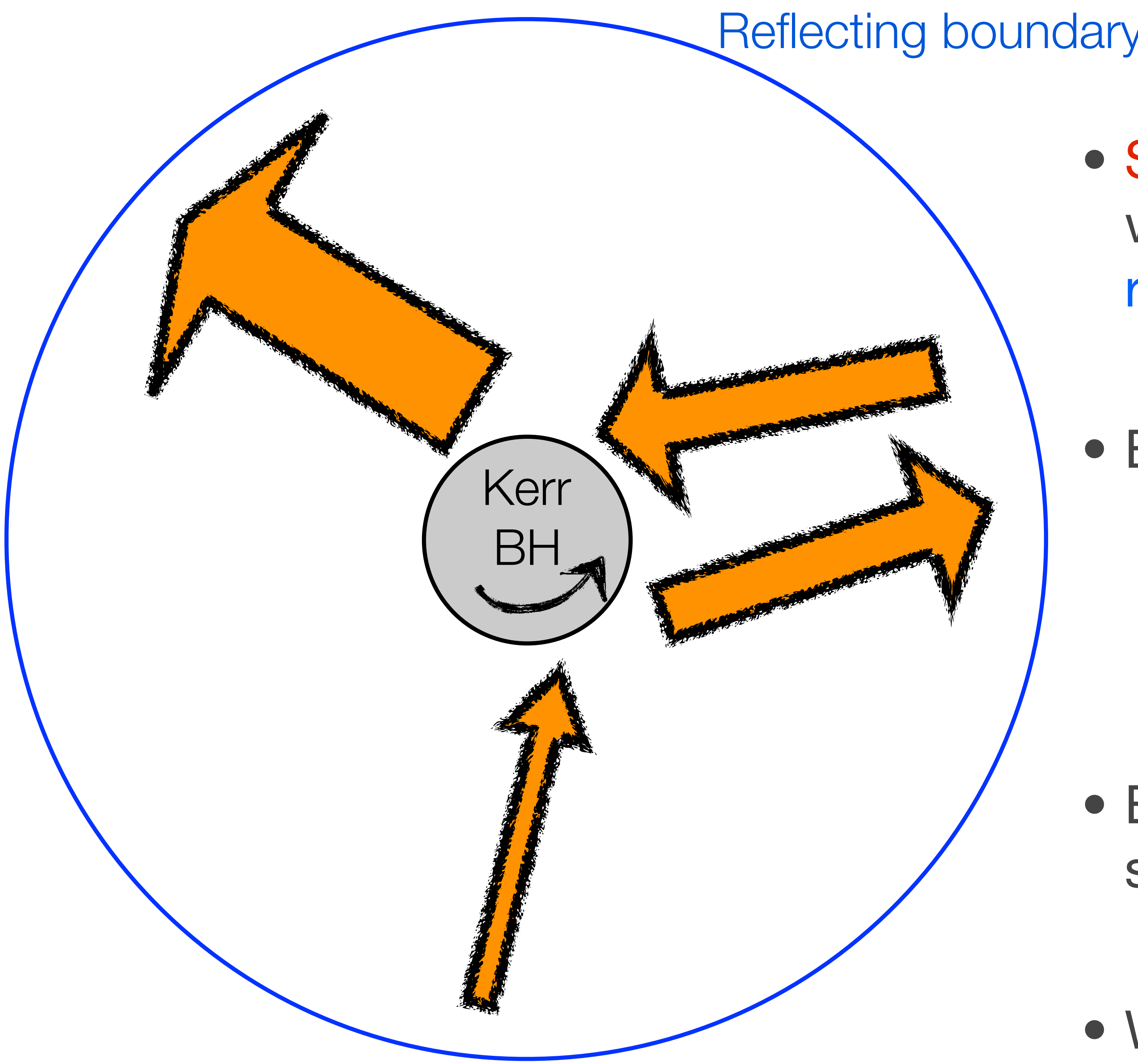


- Can be understood from the area theorem:
  - Wave $\sim e^{im\phi}e^{-i\omega t}$ changes BH area by

$$\frac{\kappa}{8\pi}\delta A = \delta M - \Omega_H \delta J$$
$$= \delta M \left(1 - \Omega_H \frac{\delta J}{\delta M}\right)$$
$$= \delta M \left(1 - \Omega_H \frac{m}{\omega}\right)$$

  - Thus, if $\boxed{0 < \omega < m\Omega_H}$, area increase requires $\delta M < 0$

Introduction to superradiant instability



- Superradiant instability caused when ergoregion combined with reflecting boundary.
- Examples:
  - mass term for field
  - mirror
  - anti-de Sitter boundary
- Black hole must be sufficiently small, or else no ergoregion
- What is the end state?

Outline

1. Linear superradiant instability of AdS black holes with ergoregions to gravitational perturbations
   - Canonical energy method of Hollands and Wald
   - Construction of unstable initial data; all such black holes unstable
2. Nonlinear evolution of superradiant instability of Reissner-Nordstrom-AdS black holes
   - Spherically symmetric numerical relativity simulations
   - Backreaction on black hole, evolution of individual modes, final state

Part 1: Linear superradiant instability

CQG **33** 125022 (2016) (arXiv:1512.02644) with S. Hollands, A. Ishibashi and R. Wald

- Background metric g_{ab}
    - asymptotically AdS black hole solution to Einstein equation in $d \geq 4$
    - horizon Killing vector field K^a
- Metric perturbation γ_{ab}
    - solution to linearized Einstein equation with reflecting AdS boundary condition
- Main result: Black hole is unstable if K^a becomes spacelike somewhere outside the black hole (i.e., there is an ergoregion).

Canonical energy method

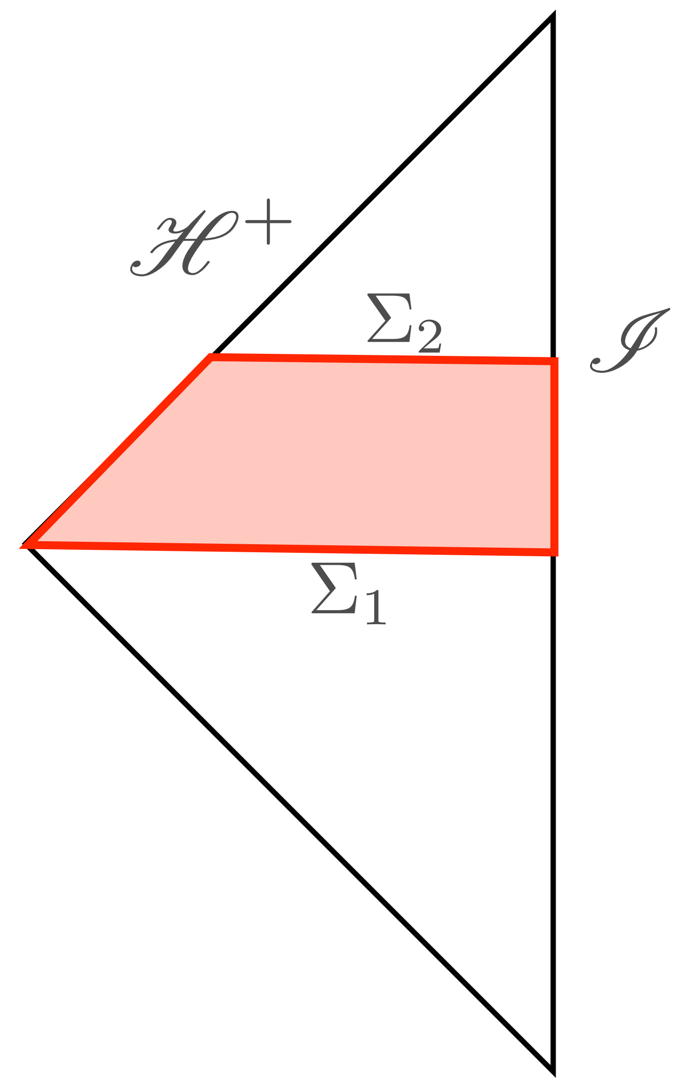
- Standard method to prove instability: Search for mode solutions that grow in time.

- This is difficult, in particular for complicated backgrounds, higher dimensions, or gravitational perturbations. Requires decoupling and separation of equations, which may not even be possible.

- Alternative is “canonical energy method”, which only requires construction of initial data solving the constraint equations---not a solution to the evolution equations.

Canonical energy method

- Canonical energy $\mathcal{E}$ is an integral over a Cauchy hypersurface Σ, quadratic in the perturbation γ_{ab}, satisfying
    - Gauge invariance
    - Degeneracy precisely on perturbations to other stationary black holes
    - Conservation
    - Positive flux at horizon and infinity



- Then $\mathcal{E}_{\Sigma_2} < \mathcal{E}_{\Sigma_1}$, and if a solution to the constraints γ_{ab} exists such that $\mathcal{E}_{\Sigma_1}(\gamma) < 0$, instability follows.

Canonical energy

- Starting with Einstein-Hilbert action, one can derive a symplectic current, which depends on two metric perturbations,

$$w^a(\gamma_1, \gamma_2) = \frac{1}{16\pi} g^{abcdef} \left(\gamma_{2bc} \nabla_d \gamma_{1ef} - \gamma_{1bc} \nabla_d \gamma_{2ef} \right),$$

  where $g^{abcdef} = g^{ae} g^{fb} g^{cd} - \frac{1}{2} g^{ad} g^{be} g^{fc} - \frac{1}{2} g^{ab} g^{cd} g^{ef} - \frac{1}{2} g^{bc} g^{ae} g^{fd} + \frac{1}{2} g^{bc} g^{ad} g^{ef}$ depends on the background metric.

- Symplectic form: $W_\Sigma(g; \gamma_1, \gamma_2) = \int_\Sigma n^a w_a$

  spacelike hypersurface (Σ)

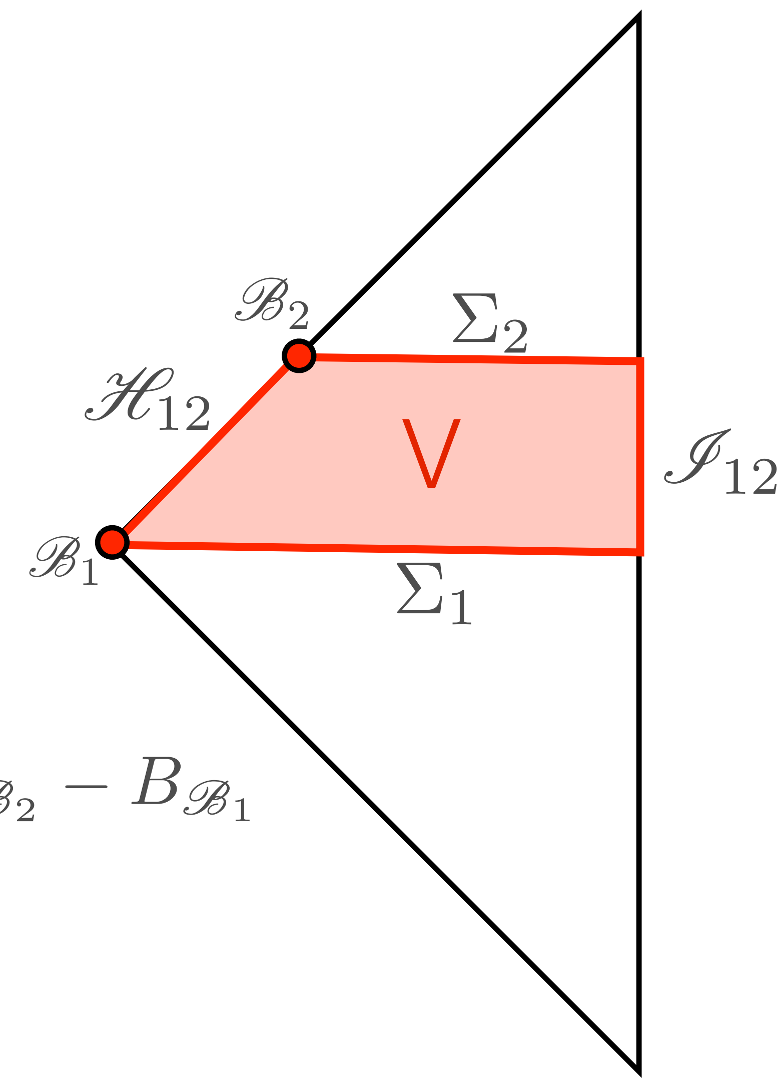
- For solutions to the linearized Einstein equation, $\nabla_a w^a = 0$

Canonical energy

- Integrate over a volume V. On solutions, Stokes' theorem gives

$$0 = \int_V \nabla_a w^a = \int_{\partial V} n_a w^a$$

- Now take $\gamma_2 = \pounds_K \gamma_1$, so $w^a = w^a(\gamma, \pounds_K \gamma)$ and consider contributions from each boundary



$$\int_{\mathscr{I}_{12}} n_a w^a = 0$$

$$\int_{\mathscr{H}_{12}} n_a w^a = \frac{1}{4\pi} \int_{\mathscr{H}_{12}} (K^c \nabla_c u) \delta\sigma_{ab} \delta\sigma^{ab} + B_{\mathscr{B}_2} - B_{\mathscr{B}_1}$$

↑ nonnegative

(imposed reflecting AdS boundary, and certain gauge conditions)

Canonical energy

- So define the canonical energy

$$\mathcal{E}_K(\gamma, \Sigma) = W_\Sigma(g; \gamma, \pounds_K \gamma) - B_{\mathscr{B}}(g; \gamma)$$

- Above implies $\mathcal{E}_K(\gamma, \Sigma_2) \leq \mathcal{E}_K(\gamma, \Sigma_1)$ (decreases in time)

- Under restriction to certain gauge conditions at $\mathscr{H}^+$ and $\mathscr{I}$, together with $\delta A = 0$ and $\delta H_X = 0$ for all asymptotic symmetries X^a, it can be shown that $\mathcal{E}_K(\gamma, \Sigma)$ is gauge-invariant and degenerate precisely on perturbations to other stationary black holes.

Stability criterion

Canonical energy non-negative for all such perturbations	Canonical energy negative for some perturbation
STABLE	**UNSTABLE**

Construction of initial data

- Energy (with respect to K^a) of a particle with 4-momentum p^a is

$$\mathcal{E}_{K,\mathrm{particle}} = -K^a p_a$$

  If there is an ergoregion where $K^a K_a > 0$ is spacelike, then a timelike or null p^a may be chosen to make $\mathcal{E}_{K,\mathrm{particle}} < 0$ in the ergoregion.

- Similarly, for a wave, we ought to be able to find a gravitational perturbation such that the canonical energy $\mathcal{E}_K(\gamma) < 0$

  - **Step 1:** WKB method to obtain approximate compact support solution to the constraint equations of the form $\gamma_{ab} = A_{ab}\exp(i\omega\chi)$ with $\omega \gg 1$ and $\mathcal{E}_K(\gamma) \sim \omega^2 K^a p_a < 0$

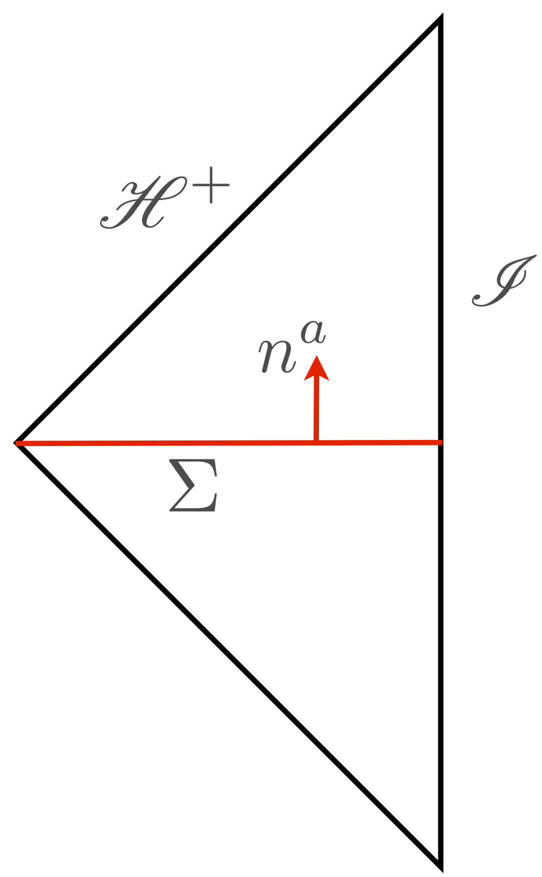
  - **Step 2:** Obtain exact solution with Corvino-Schoen method, such that canonical energy remains negative.

Construction of initial data

- Convenient to trade spacetime quantities g_{ab} and γ_{ab} for initial data quantities defined on Σ

$$q_{ab} = g_{ab} + n_a n_b$$
$$p^{ab} = \sqrt{q}(k^{ab} - q^{ab} k^c{}_c)$$

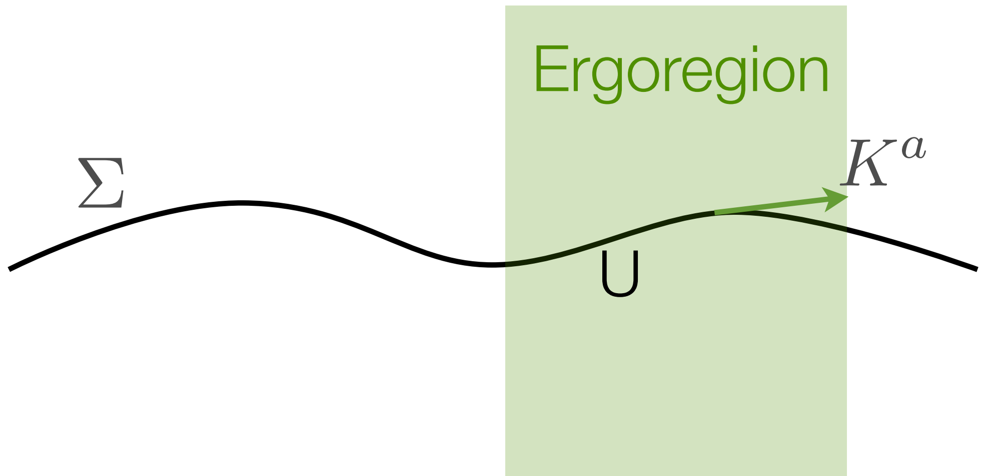
$$\delta q_{ab} = q_a{}^c q_b{}^d \gamma_{cd}$$
$$\delta p^{ab} = \sqrt{q}(q^{ac} q^{bd} - q^{ab} q^{cd}) \frac{1}{2} \pounds_n \gamma_{cd}$$



Construction of initial data

- Assume there is a region where K^a is spacelike. Construct approximate initial data of compact support in this region.

- Trick: In this region, choose Σ such that it is tangent to K^a (possible since spacelike). This leads to the expression



$$\mathcal{E}_K(\delta q_{ab}, \delta p^{ab}) = -\frac{1}{16\pi}\int_\Sigma K^a \left(-2\delta p^{bc} D_a \delta q_{bc} + 4\delta p^{cb} D_b \delta q_{ac} + 2\delta q_{ac} D_b \delta p^{cb} -2p^{cb}\delta q_{ad} D_b \delta q_c{}^d + p^{cb}\delta q_{ad} D^d \delta q_{cb}\right)$$

- Constraints

$$\boldsymbol{C}(\delta q_{ab}, \delta p^{ab}) \equiv \begin{pmatrix} q^{\frac{1}{2}}\left(D^a D_a \delta q_c{}^c - D^a D^b \delta q_{ab} + Ric(q)^{ab}\delta q_{ab}\right) + \\ q^{-\frac{1}{2}}\left(-\delta q_c{}^c p^{ab} p_{ab} + 2p_{ab}\delta p^{ab} + 2p^{ac}p^b{}_a \delta q_{bc} + \right. \\ \left. \frac{1}{d-2}p^c{}_c p^d{}_d \delta q^a{}_a - \frac{2}{d-2}p^a{}_a \delta p^b{}_b - \frac{2}{d-2}\delta q_{ab} p^{ab} p_c{}^c\right) \\ \\ -2q^{\frac{1}{2}} D^b(q^{-\frac{1}{2}}\delta p_{ab}) + D_a \delta q_{cb} p^{cb} - 2D_c \delta q_{ab} p^{bc} \end{pmatrix} = 0$$

Construction of initial data

- WKB expansion of initial data

$$\delta q_{ab} = \left(\sum_{n \geq 0} Q_{ab}^{(n)} (i\omega)^{-n} \right) \exp(i\omega\chi),$$

$$\delta p_{ab} = \left(\sum_{n \geq 0} P_{ab}^{(n)} (i\omega)^{-n+1} \right) \exp(i\omega\chi)$$

WKB parameter

phase function

- Constraints become

$$\begin{pmatrix} -D^a\chi(D_a\chi)Q_c^{(n)c} + D^a\chi(D^b\chi)Q_{ab}^{(n)} \\ P_{ab}^{(n)} D^b\chi \end{pmatrix} = \boldsymbol{C}^{(n)}$$

Depends on lower order (m<n)
WKB approximations

  - <u>0th order</u>, choose

$$P_{ab}^{(0)} = -Q_{ab}^{(0)}, \qquad Q_a^{(0)a} = 0, \qquad Q_{ab}^{(0)} D^b\chi = 0$$

  - <u>Higher orders</u> algebraic

Construction of initial data

- To leading order in WKB, the canonical energy is

$$\mathcal{E}(\delta q, \delta p) = -\frac{\omega^2}{16\pi}\int_U K^b D_b \chi \, Q_c^{(0)a} Q_a^{(0)c} + O(\omega)$$

- So choosing $K^a D_a \chi > 0$ gives $\mathcal{E} < 0$ as $\omega \to \infty$

- Of course, any given WKB order is only an approximate solution. Using the Corvino-Shoen method (see paper), we can correct our WKB initial data such that

    - Linearized constraints hold exactly

    - Data remain smooth and compactly supported in slightly larger region

    - The correction to the canonical energy is sufficiently small as $\omega \to \infty$

Conclusions from part 1

- *Any black hole in AdS with a horizon Killing field that becomes spacelike is linearly unstable to superradiant gravitational perturbations.* Results follow from a Lagrangian formulation of the theory, so should carry over to other fields.

- As perturbation grows, nonlinear effects become important:

  - Backreaction of the perturbation on the black hole changes the background

  - Changing background alters the dynamics of the perturbation. Unstable modes may become stable.

- What is the end point of the instability? Speculation includes violation of cosmic censorship, as there is no plausible stable final state. Numerical simulations are important, but challenging.

Part 2: RN-AdS superradiant instability

PRL **116** 141102 (2016) (arXiv:1601.0138) with P. Bosch and L. Lehner

- Reissner-Nordstrom-AdS black holes are also subject to the superradiant instability, with charge playing the role of angular momentum.

- Charged scalar field mode $\psi \sim e^{-i\omega t}$ superradiantly amplified if

$$\omega r_H < qQ$$

Black hole radius (r_H)

gauge coupling (q)

Black hole charge (Q)

compare rotating case: $\omega < m\Omega_H$

- *Instability occurs even in spherical symmetry, which makes numerical simulations feasible.*

Model

- Fields: g_{ab} – metric

  A_a – Maxwell

  ψ – complex scalar

- Lagrangian: $$16\pi G_N \mathcal{L} = R + \frac{6}{L^2} - \frac{1}{4} F_{ab} F^{ab} - |D_a \psi|^2$$

  $$D_a = \nabla_a - iqA_a$$

  This gives rise to the Einstein, Maxwell, and scalar field equations, which we solve numerically.

- It can be checked that RN-AdS is a solution

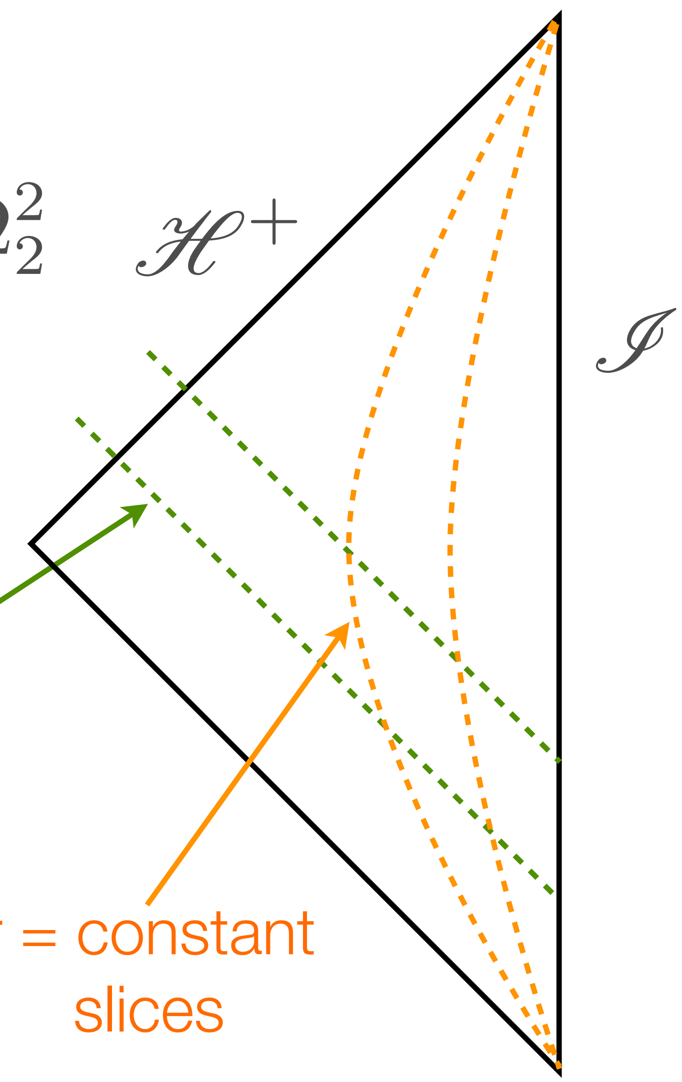
Numerical method

- We work in Eddington-Finkelstein coordinates and spherical symmetry. Metric and Maxwell fields can be put in the form

$$ds^2 = -A(v,r)dv^2 + 2dvdr + \Sigma(v,r)^2 d\Omega_2^2$$

$$A_\mu dx^\mu = W(v,r)dv$$

$$\psi = \psi(v,r)$$



- Reflecting boundary conditions at $\mathscr{I}$ that fix ADM mass M, charge Q.

- Initial data $\psi(v = v_0, r)$

Numerical method

- Equations of motion are highly coupled

Einstein:

$$0 = \Sigma(d_+\Sigma)' + (d_+\Sigma)\Sigma' - \frac{3}{2L^2}\Sigma^2 - \frac{1}{2} + \frac{1}{8}\Sigma^2 W',$$

$$0 = A'' - \frac{4}{\Sigma^2}(d_+\Sigma)\Sigma' + \frac{2}{\Sigma^2} + (\psi')^* d_+\psi$$
$$+ \; (d_+\psi)^*\psi' - (W')^2 + iqW\left[\psi^*\psi' - (\psi')^*\psi\right],$$

$$0 = d_+d_+\Sigma - \frac{1}{2}A'd_+\Sigma + \frac{1}{2}\Sigma|d_+\psi|^2 + \frac{1}{2}q^2W^2\Sigma|\psi|^2$$
$$+ \; \frac{1}{2}iqW\Sigma\left[\psi^* d_+\psi - \psi(d_+\psi)^*\right],$$

$$0 = \Sigma'' + \frac{1}{2}\Sigma|\psi'|^2$$

Maxwell:

$$0 = (d_+W)' - \frac{1}{2}A'W' + 2\frac{d_+\Sigma}{\Sigma}W' - 2q^2W|\psi|^2$$
$$+ \; iq\left(\psi^* d_+\psi - \psi(d_+\psi)^*\right],$$

$$0 = W'' + \frac{2}{\Sigma}\Sigma'W' + iq\left[\psi^*\psi' - \psi(\psi')^*\right]$$

Scalar:

$$0 = 2(d_+\psi)' + 2\frac{\Sigma'}{\Sigma}d_+\psi + 2\frac{d_+\Sigma}{\Sigma}\psi' - iq\psi W'$$
$$- \; 2iq\frac{\Sigma'}{\Sigma}W\psi - 2iqW\psi'$$

$$\text{where} \quad f' \equiv \partial_r f$$
$$d_+ f \equiv \partial_v f + \frac{1}{2}A\partial_r f$$

Numerical method

- Asymptotically near r = infinity, obtain a power series solution

$$A = \frac{r^2}{L^2} + 1 - \frac{2M}{r} + \frac{Q^2}{4r^2} + O(r^{-3}),$$
$$\Sigma = r + O(r^{-5}),$$
$$W = \frac{Q}{r} + O(r^{-2}),$$
$$\psi = \frac{\varphi_3(v)}{r^3} + O(r^{-4})$$

where $M = \text{ADM mass}$

$Q = \text{ADM charge}$

$\varphi_3(v) = \text{unknown function, determined from solution}$

- This imposes reflecting boundary conditions at infinity, and also fixes a residual gauge freedom.

- M and Q are chosen and set as boundary conditions.

Numerical method

- Integration procedure

initial data: $\psi(v = v_0, r)$

Repeat

Integrate equations, radially inward in r
Impose M and Q as boundary conditions

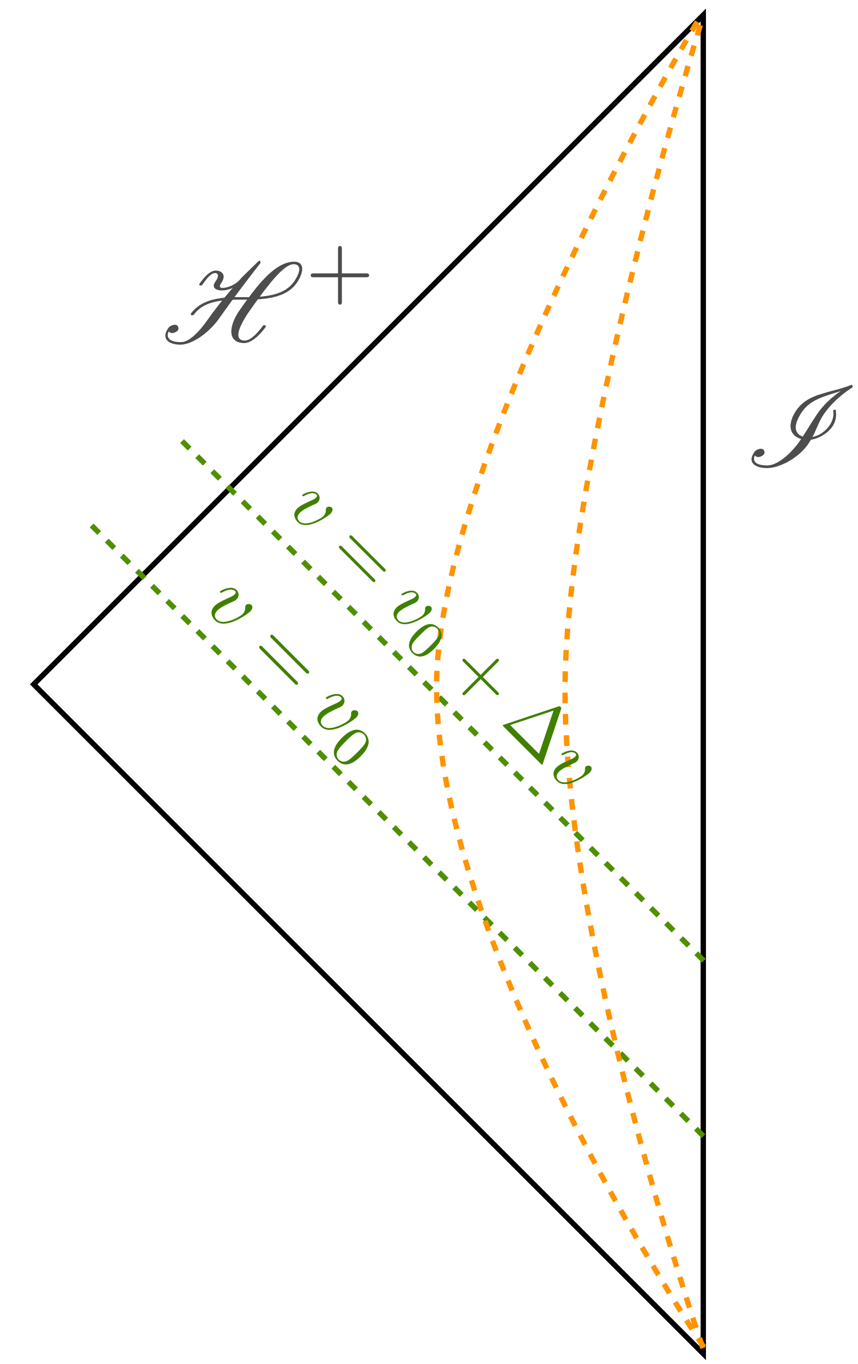
$d_+\Sigma, A, \Sigma, W, \text{and } d_+\psi \text{ at } v = v_0$

$$\partial_v \psi = d_+\psi - \frac{1}{2} A \partial_r \psi$$

$\partial_v \psi(v = v_0, r)$

Integrate one step in v

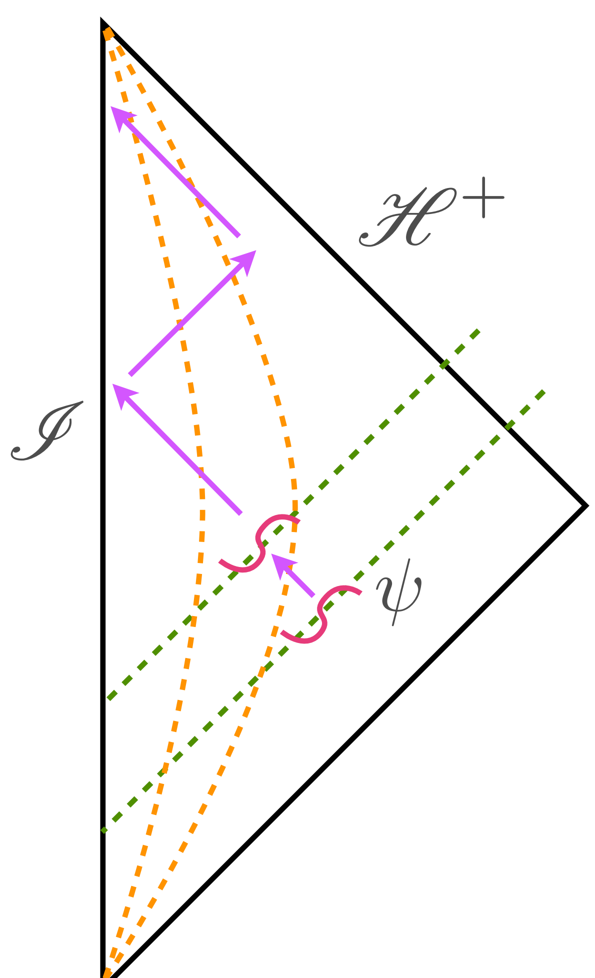
$\psi(v = v_0 + \Delta v, r)$



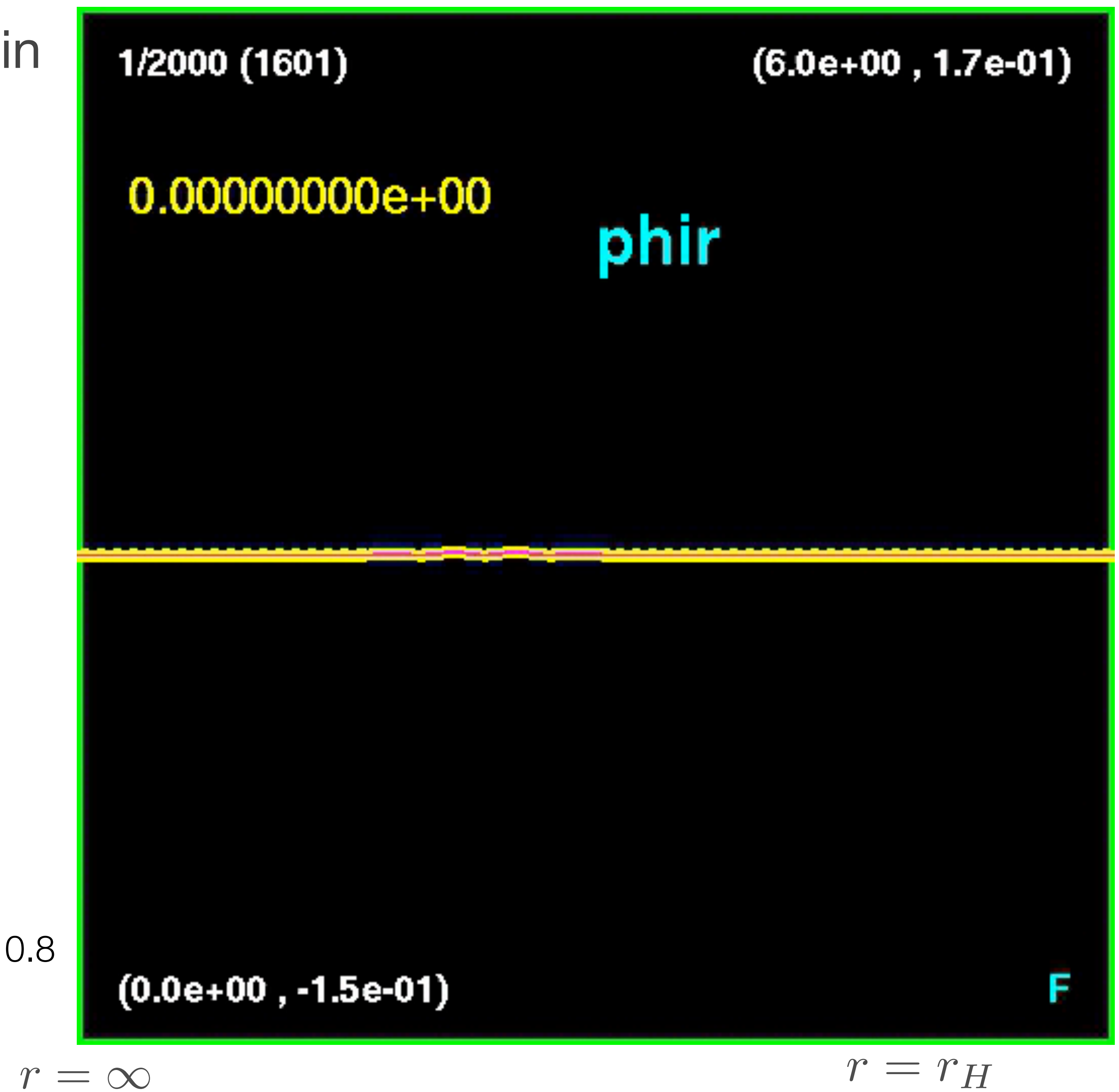
Sample evolution

- We consider small black holes in AdS, so $r_H \ll L$

- Compactly supported initial data for ψ, small amplitude.

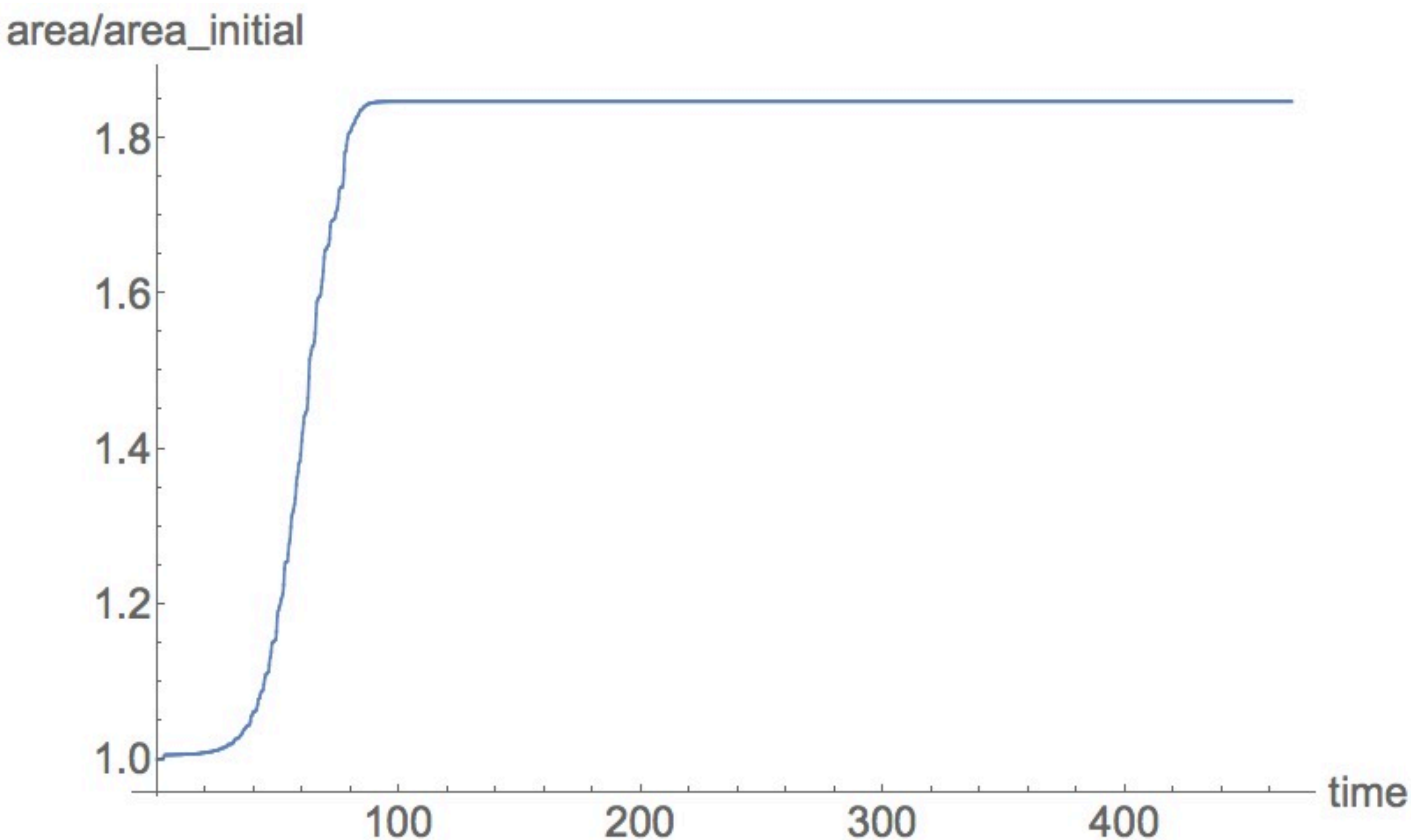


rH = 0.2
L = 1
Q/Qmax = 0.8
q = 12



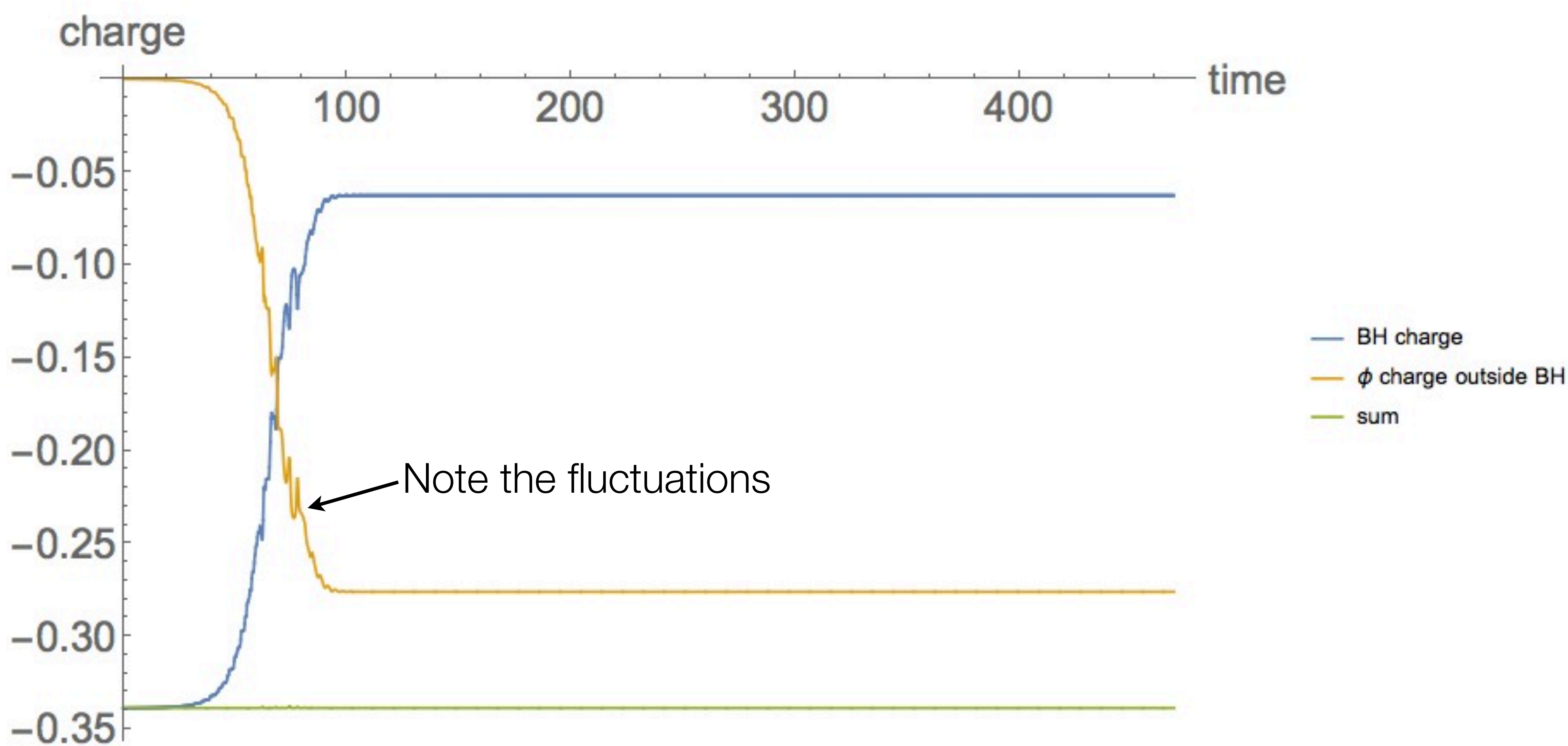
Apparent horizon area vs time

- Area always increases

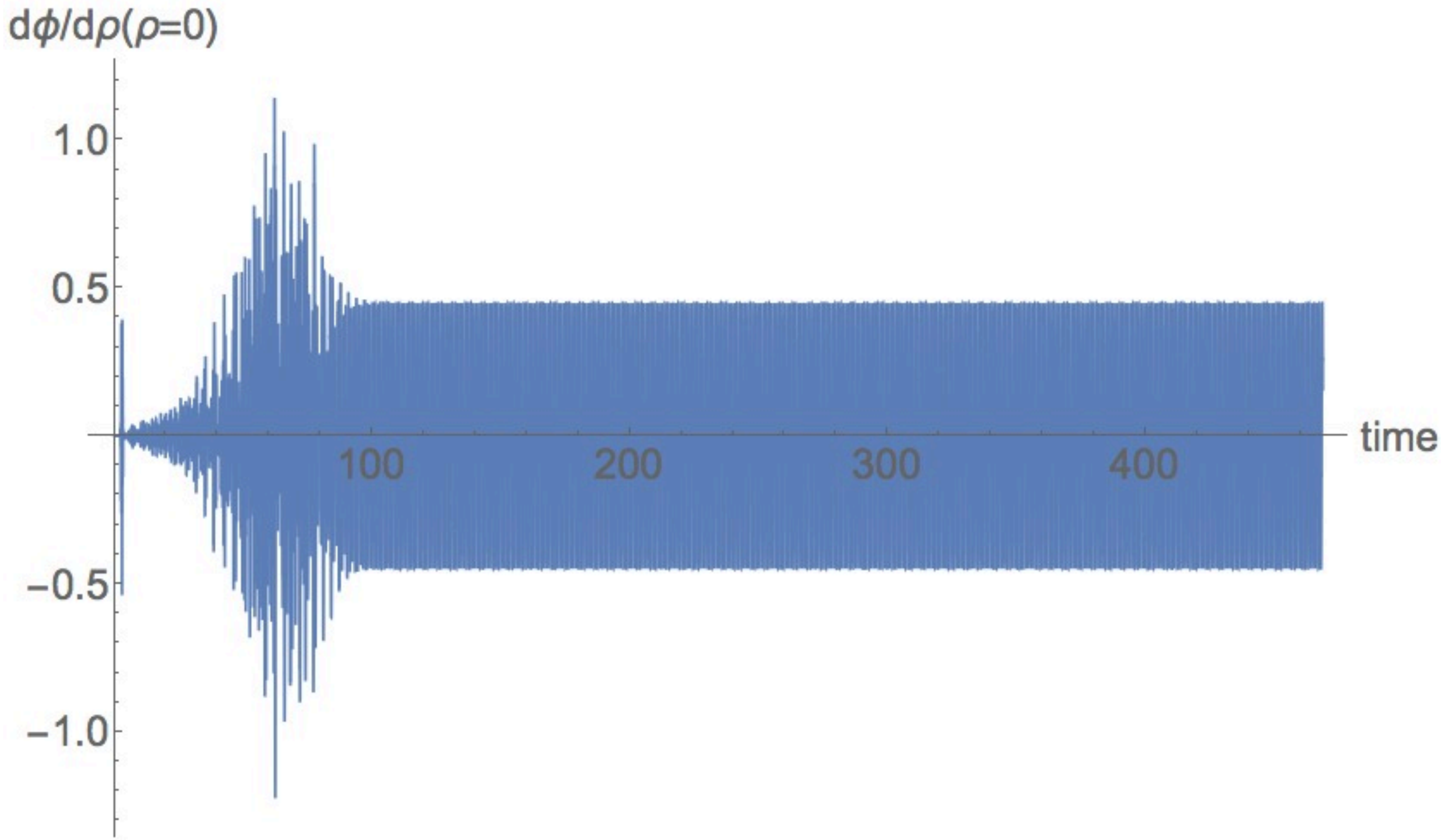


Charge vs time

- Most of the charge is extracted by the scalar field

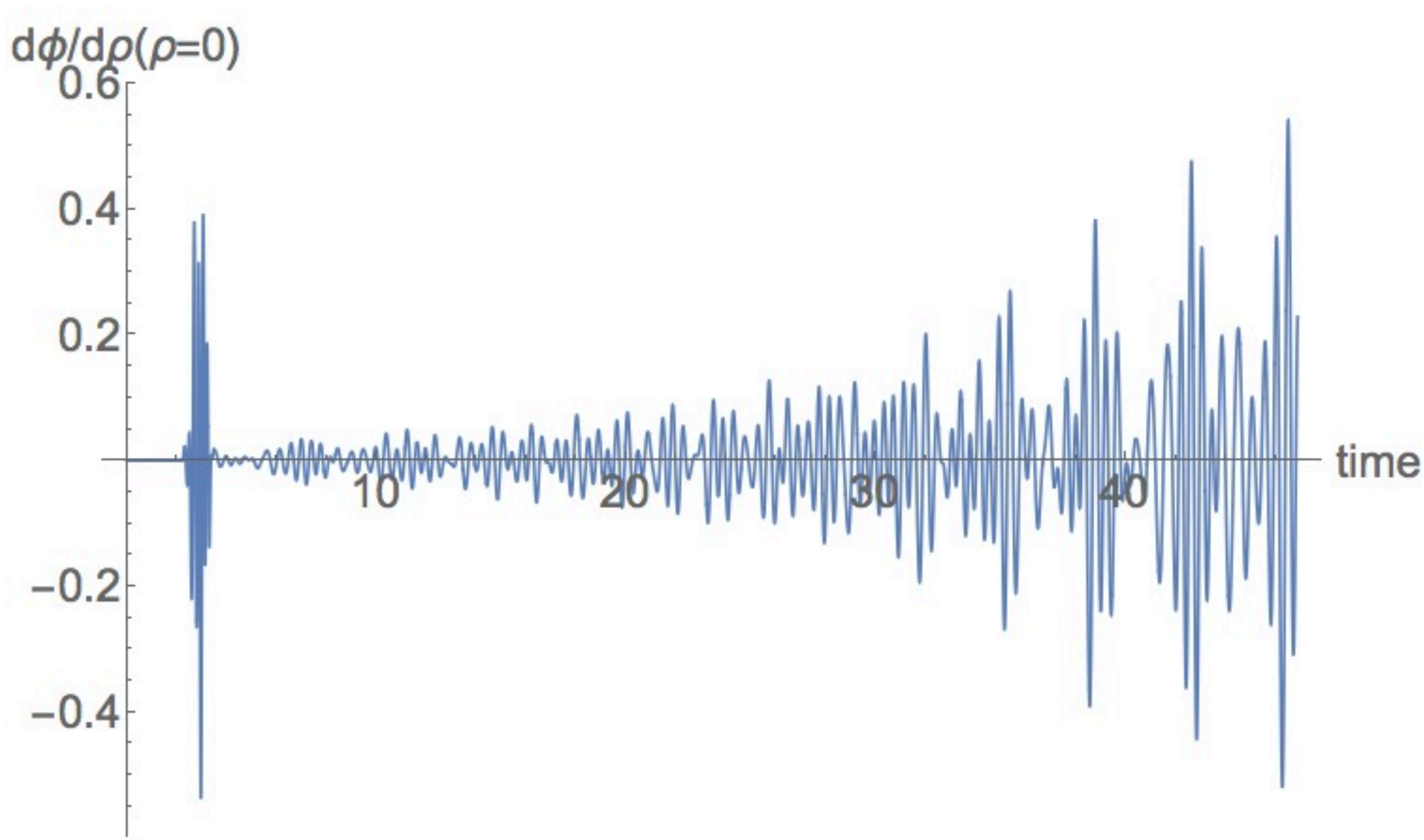


Boundary field $\varphi_3(v)$



Boundary field $\varphi_3(v)$

- Zoomed in at early times, growth clearly isn't in a single mode



Scalar field modes

- Since the black hole is small compared to AdS scale, we can approximate the scalar field modes by the empty AdS modes

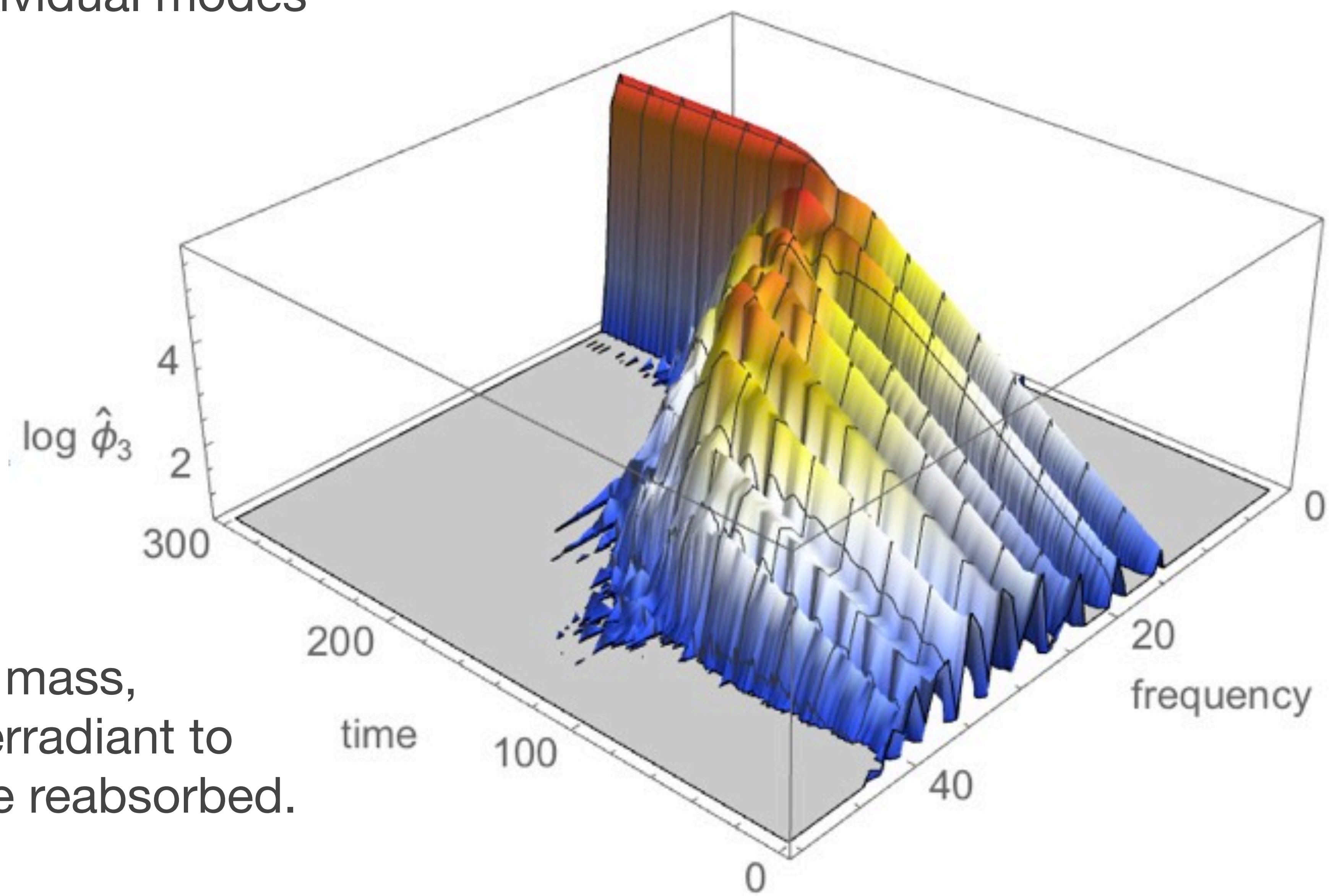
$$\omega_n \approx \frac{2n+3}{L}, \quad n = 0, 1, 2, \ldots$$

- Instability criterion $\omega r_H < qQ \implies \boxed{2n+3 < \frac{qQL}{r_H}}$

- Thus there can be several modes, and n=0 is most unstable.

Scalar field modes

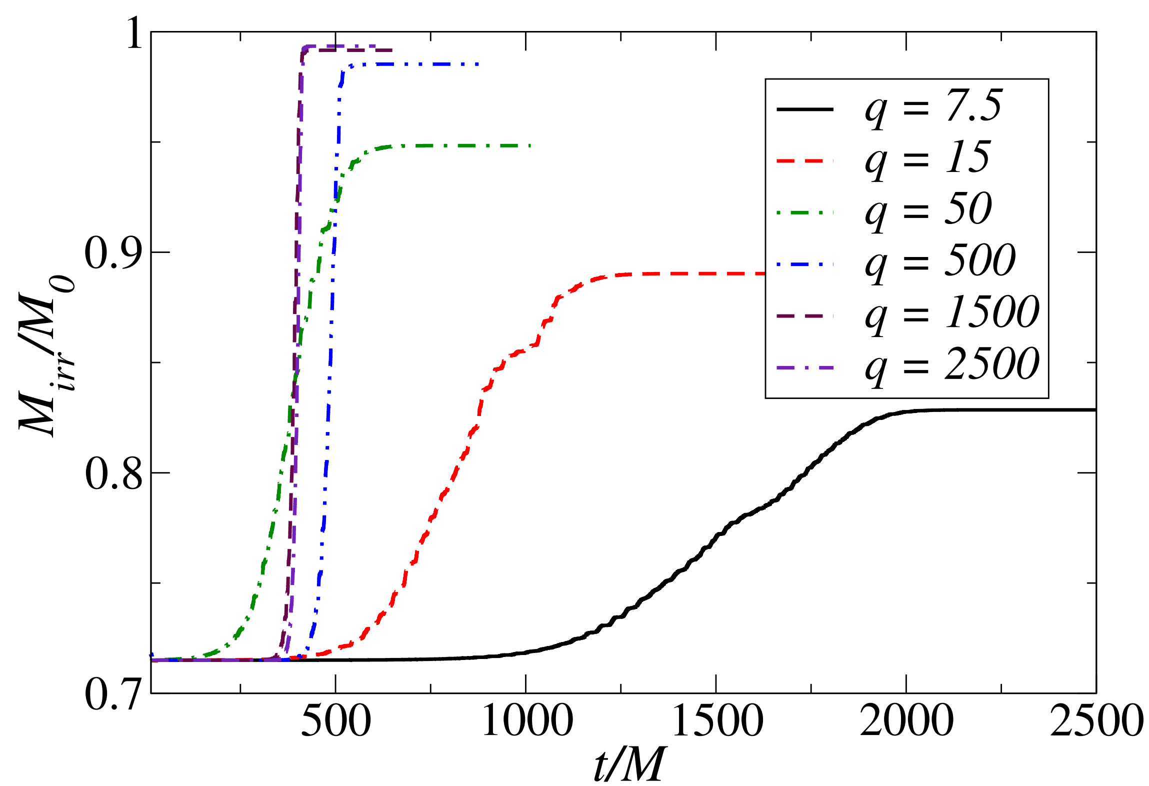
- Spectrogram reveals individual modes

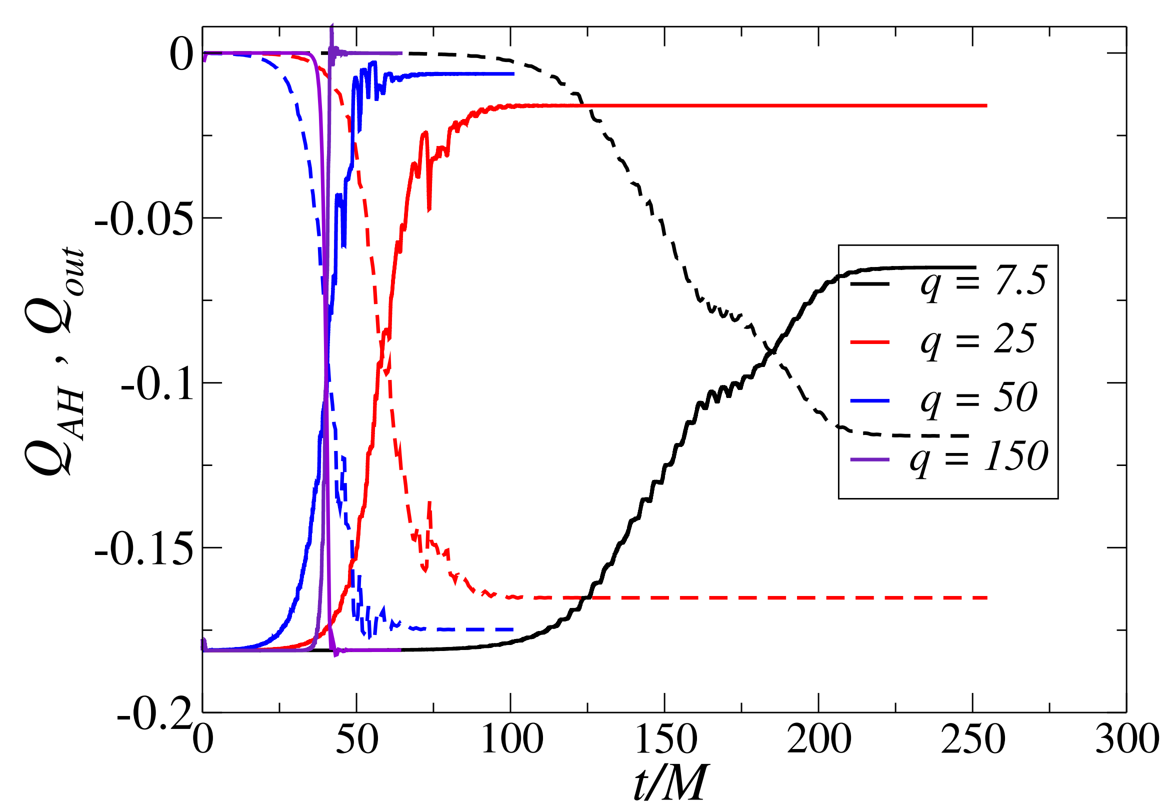


- As BH loses charge and mass, modes switch from superradiant to nonsuperradiant, and are reabsorbed.
- Final state is BH + lowest mode, with zero growth rate.

What happens when q is increased?

- Larger q excites more modes -> faster process. Recall $2n + 3 < \frac{qQL}{r_H}$

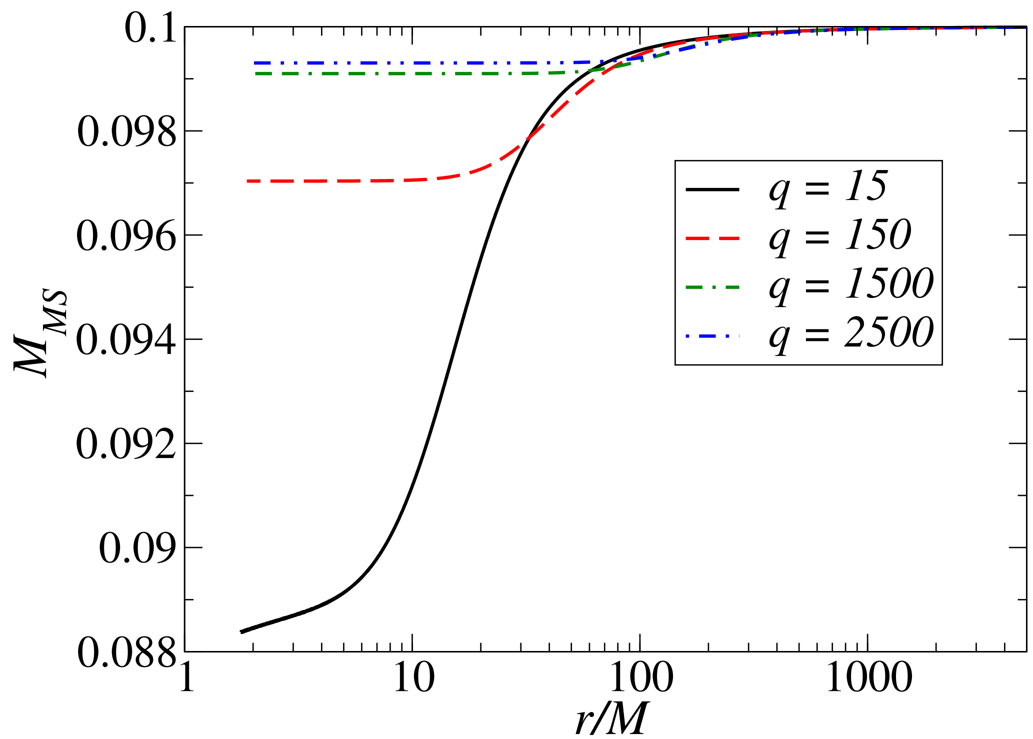




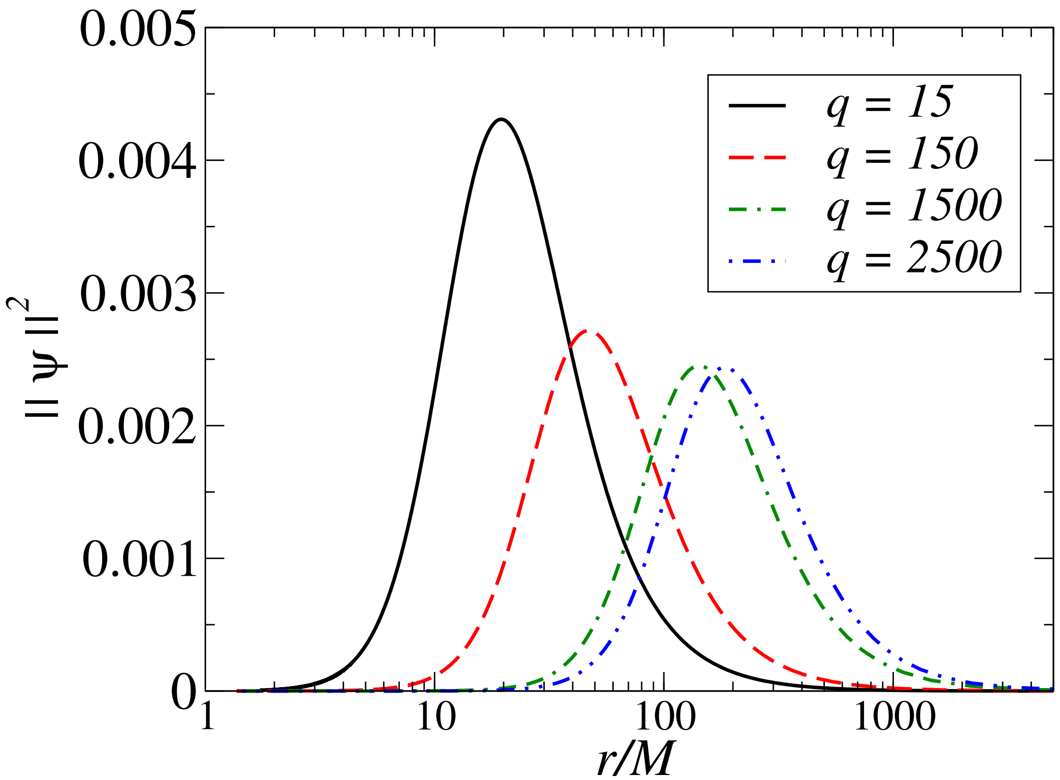
- Larger q extracts more charge

What happens when q is increased?

- End state



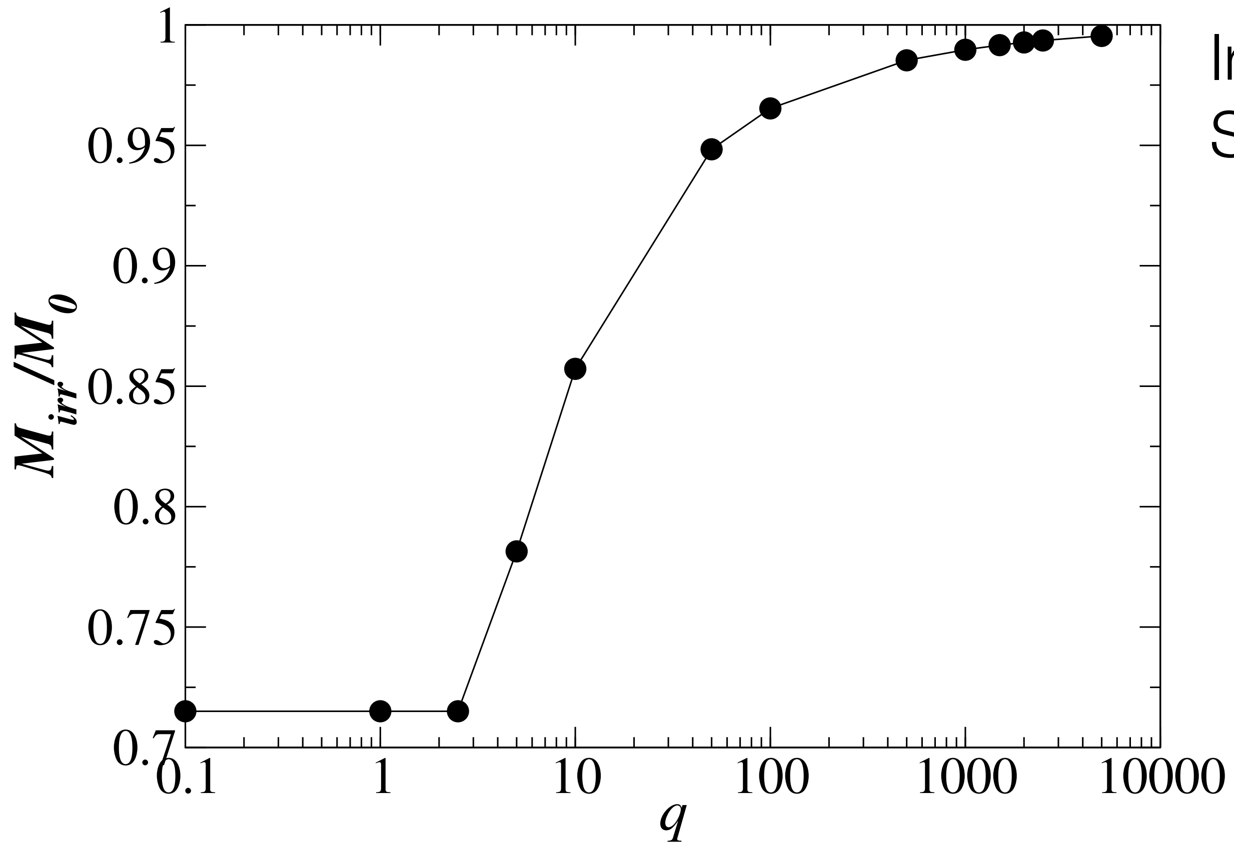
Charge is located further away from BH



Scalar field settles down further away

What happens when q is increased?

- End state



Irreducible mass (area) approaches that of a Schwarzschild-AdS BH of mass M

- *For larger q, the scalar field charge/mass ratio is increased. As the scalar field extracts nearly the full ADM charge Q, it extracts very little mass. Final state approaches Schwarzschild-AdS, surrounded by a distant low-mass/high-charge condensate.*

Conclusions

- Rotating case?

	RN-AdS	**Kerr-AdS**
Instability criterion (for a given ω)	$\omega < \frac{qQ}{r_H}$	$\omega < m\Omega_H$
	q = fixed parameter	m = any integer
Most unstable mode (final state) $\omega \approx \frac{2n+l+3}{L}$	$n = 0$ $l = 0$	$n = 0$ $l = m \to \infty$
BH instability criterion	$\frac{3}{L} < \frac{qQ}{r_H}$	$\frac{1}{L} < \Omega_H$ (Hawking-Reall bound)

Conclusions

- At the linear level, all AdS black holes with ergoregions are unstable.
- RN-AdS case: Numerical simulations show that charge and mass is extracted from the black hole by several superradiant scalar field modes. As this unfolds, higher-frequency modes cease to be superradiant, and fall back into the black hole, resulting in nontrivial dynamics. Final state is a stable hairy black hole, with the scalar condensate distributed far away for large q.
- Kerr-AdS case: The same arguments suggest an $m \to \infty$ condensate in the final state, since this is the most superradiant mode.
- Astrophysics: Finite-sized barrier arises from a mass term (no longer infinite) provides a cutoff in mode energy that can be confined.

THANK YOU