Local and Global Dynamics in Asymptotically Anti-de Sitter spaces

(parts joint with J. Luk, A. Shao, J. Smulevici, C. Warnick)

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Overview

1. Stability Problems in aAdS (mostly on black hole backgrounds)
   (a) linear Klein-Gordon equation $\Box_g \psi + \alpha \psi = 0$
   (b) spherically symmetric Einstein-Klein-Gordon system
2. Unique Continuation Properties of aAdS
   (a) the Klein-Gordon equation $\Box_g \psi + \alpha \psi = 0$
   (b) the Einstein equations (work in progress)

Anti-de Sitter space

Maximally symmetric solution of the vacuum Einstein equations,

$$Ric\,[g] = \Lambda g$$

with $\Lambda < 0$. We work in $3+1$ dimensions; choose $\Lambda = -\frac{3}{l^2} = -3$.

The AdS-metric can be written in global polar coordinates on $\mathbb{R}^4$ as

$$g_{AdS} = -\left(1+r^2\right)dt^2 + \left(1+r^2\right)^{-1}dr^2 + r^2 d\Omega^2_{\mathbb{S}^2}\ .$$

Metric is conformal to part of the ESU $(r = \tan\psi,\ \psi \in (0, \pi/2))$,

$$g_{AdS} = \frac{1}{\cos^2\psi}\left(-dt^2 + d\psi^2 + \sin^2\psi d\Omega^2_{\mathbb{S}^2}\right)\ .$$

In the conformal picture think of cylinder with spatial slices being a hemisphere of $\mathbb{S}^3$, the equator being the boundary at infinity.

Draw Penrose diagram and compare with Minkowski.

- AdS is not globally hyperbolic
- AdS is geodesically complete. Null geodesics refocus!

We also recall, the famous Schwarzschild-AdS and Kerr-AdS spacetimes which will play an important role.

From a PDE point of view, we would like to understand the dynamics near these solutions.

Einstein equations are non-linear wave equations. A prerequisite is understanding *linear* equations on fixed asymptotically AdS backgrounds.

Local dynamics (well-posedness) in AdS

Start with *linear* hyperbolic equations on fixed (asymp.) AdS.

1. (W)ave equation: $\Box_{g_{AdS}} u + \alpha u = 0 \quad \alpha < 9/4$
2. (M)axwell's equations $dF = 0$ and $d \star_{g_{AdS}} F = 0$
3. (B)ianchi equations $\nabla^a W_{abcd} = 0$ for a Weyl field W.

(W) with $\alpha = 2$ is the conformally coupled case:

$$\Box_{g_{AdS}} u + 2u = 0 \quad \Leftrightarrow \quad \Box_{g_{ESU}} v - v = 0 \text{ for } v = \left(\frac{u}{\cos\psi}\right)$$

Not globally hyperbolic $\rightarrow$ boundary conditions required

For the "finite" problem:
$v \rightarrow 0$ (Dirichlet) and $\partial_\psi v \rightarrow 0$ (Neumann)

Local dynamics (well-posedness) in AdS

For general mass, it is already non-trivial to *state* the boundary conditions correctly. Set $\kappa = \sqrt{9/4 - \alpha}$

$$r^{3/2-\kappa} u \to 0 \text{ (Dir)} \quad \text{and} \quad r^{1+2\kappa} \partial_r \left(r^{3/2-\kappa} u \right) \to 0 \text{ (Neu)}$$

Reformulate the problem in terms of twisted derivatives (Warnick).

Theorem 1. *The equations (W), (M) and (B) are well-posed on asymptotically AdS spacetimes for either Dirichlet-, Neumann- or dissipative boundary conditions.*

$\to$ Breitenlohner-Freedman, Bachelot, Vasy, G.H., Warnick, ...

Local dynamics (well-posedness) in AdS

I haven't told you yet about dissipative conditions. Boundary conditions deal with the term on $\mathcal{I}$ in the "usual" energy estimate.

$$\int_{t_1}^{t_2} dt \int_{S^2} \sin\theta d\theta d\phi \; \partial_\psi v \partial_t v \qquad \text{with} \;\; v = \frac{u}{\cos\psi}$$

(Dir) and (Neu) make this term vanish, dissipative conditions give it a sign:

$$\partial_\psi v = -\partial_t v \quad \text{on } \mathcal{I}$$

$\rightarrow$ energy leaving the spacetime through the boundary.

- needs care if $\alpha \neq 2$
- For (M) and (B): null-decomposition to isolate radiation fields. For (B) need to introduce reduced system (Friedrich, 1986).

Non-linear Einstein-Klein Gordon system

Once the linear problem is understood, one may study a *non-linear* toy-model.

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} &= 8\pi T_{\mu\nu} \\
T_{\mu\nu} &= \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \left(\partial_\beta \phi \partial^\beta \phi - \frac{\alpha}{l^2} \phi^2 \right) \\
\Box_g \phi + \frac{\alpha}{l^2} \phi &= 0
\end{aligned}
$$

In **spherical symmetry** local double-null coordinates

$$
g = -\Omega^2 (u, v)\, du dv + r^2 (u, v)\, d\omega_{\mathbb{S}^2} \tag{1}
$$

We obtain a non-linear system of $1+1$ dimensional PDEs for Ω, r, ϕ. The dynamics is "governed" by ϕ due to Birkhoff's theorem ($\phi = 0 \rightarrow$ solution has to be a member of the Schwarzschild-AdS family)

Within this model one may want to understand issues like weak cosmic censorship and black hole formation just as in the asymptotically-flat case (cf. Christodoulou). The following has been carried out:

Theorem 2. *(G.H., J. Smulevici) The EKG system is well-posed in appropriate function spaces for Dirichlet conditions on* ϕ*.*

Theorem 3. *(G.H., C. Warnick) The EKG system is well-posed in appropriate function spaces for Neumann conditions on* ϕ*.*

The system of equations

$$
\begin{aligned}
\partial_u\left(\frac{r_u}{\Omega^2}\right) &= -4\pi r\frac{(\partial_u\phi)^2}{\Omega^2}, && (2)\\
\partial_v\left(\frac{r_v}{\Omega^2}\right) &= -4\pi r\frac{(\partial_v\phi)^2}{\Omega^2}, && (3)\\
r_{uv} &= -\frac{\Omega^2}{4r}-\frac{r_u r_v}{r}+4\pi r\left(\frac{a\Omega^2\phi^2}{2l^2}\right)-\frac{3}{4}\frac{r}{l^2}\Omega^2, && (4)\\
(\log\Omega)_{uv} &= \frac{\Omega^2}{4r^2}+\frac{r_u r_v}{r^2}-4\pi\partial_u\phi\partial_v\phi, && (5)\\
\Box_{r,\Omega}\phi+\frac{\alpha}{l^2}\phi &= 0 && (6)
\end{aligned}
$$

For the WP-theorems one actually introduces a renormalized system. Then use a fixed point argument, combining energy estimates for the wave equation for ϕ with pointwise estimates for the renormalized variables r and Ω.

The theorem also provides an explicit construction of the initial data from a free function ϕ on a null-hypersurface $v = const$. Moreover, there is a notion of a maximum development.

For the second theorem (Neumann conditions), the Hawking mass needs to be renormalised (by ϕ) as it diverges at the boundary.

Once one has a local theory, one can ask global questions.

Indeed, the above model was used by *Bizon-Rostworowski* to study the instability of pure AdS (Dirichlet conditions).

Moreover, one has

Theorem 4. *(G.H., J. Smulevici) The Schwarzschild-AdS spacetime is asymptotically stable within the spherically symmetric EKG system for Dirichlet conditions on* ϕ*.*

Global problems

1. Is AdS stable? $\rightarrow$ Do linear fields on AdS decay?
2. Is Kerr-AdS stable? $\rightarrow$ Do linear fields on Kerr-AdS decay?

The answers depend on the boundary conditions.

For (Dir) and (Neu) in (W) $\rightarrow$ periodic solutions; no decay
Non-linear problem is wide open. (Numerics and heuristics: Bizon-Rostworowski, Horowitz, Santos, Lehner, Liebling, Maliborski, Buchel, Craps et al)

Remainder of this talk

1. (W), (M) and (B) for dissipative conditions
2. linear wave equation on Kerr-AdS with Dirichlet (Neumann) conditions

Let us start with the second problem:

Theorem 5 (G.H.-Smulevici)**.** *Solutions to* $\Box_{g_{M,a}}\psi + \alpha\psi = 0$ *with (Dir) decay logarithmically in time (and generally not faster) on Kerr-AdS* $g_{M,a}$ *satisfying the Hawking-Reall bound.*

Theorem 5 should be compared with the AF-case, where Dafermos–Rodnianski–Shlapentokh-Rothman proved *polynomial decay* in time. Here we expect non-linear instability.

To give you a better idea of what is actually proven, here are the estimates. They arise through energy identities/ inequalities generated by suitable vectorfields.

1. The solutions arising from data prescribed on Σ_0 remain uniformly bounded, provided $r^2_{hoz} > |a|l$ holds:

$$\sum_{i=1}^{k} \int_{\Sigma_{t^\star}} |D^i \psi|^2 \leq \sum_{i=1}^{k} \int_{\Sigma_0} |D^i \psi|^2 \quad \text{for any } k \geq 1\,.$$

2. The solutions actually satisfy for $t^\star \geq 2$

$$\int_{\Sigma_{t^\star}} |D\psi|^2 \leq \frac{C}{(\log t^\star)^2} \int_{\Sigma_0} |D^2\psi|^2 + |D\psi|^2$$

Comments on the proof

The Hawking-Reall bound $r_+^2 \geq |a|\ell$ ensures the existence of a *globally causal* Killing field on the black hole exterior $K = \partial_t + \frac{a}{r_+^2+a^2}\partial_\phi$. This gives rise to an energy estimate. Coupled with the redshift $\rightarrow$ boundedness statement.

The reason for the slow decay is a coupling of the trapping effect with the reflecting boundary conditions [G.H.-Smulevici].

The result is sharp in the sense that in particular, you can disprove any estimate of the form

$$\int_{\Sigma_t \cap \{r \geq r_{max}\}} |D\psi|^2 \leq \frac{C(t)}{(\log t)^2} \int_{\Sigma_0} |D^2\psi|^2 + |D\psi|^2$$

for a $C(t)$ with $C(t) \to 0$ as $t \to \infty$.
Note that the left hand side is the energy to the right of the potential.

Proof proceeds by construction of quasimodes (time-periodic approximate solutions to the wave equation with exponentially small errors in ℓ).

The idea of the proof (explained for Schwarzschild): Decompose

$$\psi = \psi_{\ell \leq L} + \psi_{\ell \geq L} = \sum_{\ell \leq L, m} \psi_{m\ell} Y_{m\ell} + \sum_{\ell \geq L, m} \psi_{m\ell} Y_{m\ell}$$

For $\psi_{\ell \leq L}$ prove (key: vectorfield estimate)

$$\int_{\Sigma_{t^\star}} |D\psi_{\ell \leq L}|^2 \leq C \exp\left(-e^{-L} \cdot t^\star\right) \int_{\Sigma_0} |D\psi|^2$$

For $\psi_{\ell \geq L}$ we use the boundedness statement

$$\int_{\Sigma_{t^\star}} |D\psi_{\ell \geq L}|^2 = \frac{1}{L\,(L+1)} \int_{\Sigma_{t^\star}} |\nabla\!\!\!/\, D\psi_{\ell \geq L}|^2 \leq \frac{1}{L^2} \int_{\Sigma_0} |D^2\psi|^2$$

The two estimates hold for any L. We interpolate using

$$L = \frac{1}{2} \log t^\star$$

which proves the result.

Final Remarks

See work of Warnick and Gannot on QNM in this context.

Toroidal Schwarzschild-AdS: Log-decay disappears (J. Dunn).

Hawking-Reall bound violated $\rightarrow$ superradiant instabilities appear

rigorous construction of an exponentially growing mode solution in that regime due to D. Dold (2015)

More in the talk of Stephen Green (study of superradiant instabilities using the canonical energy).

Let us turn to Problem 1, the wave equation (Maxwell, Bianchi) with dissipative boundary conditions on pure AdS.

This is joint work with Luk, Smulevici and Warnick (HLSW).

Informally, the result will be that now solutions do decay but there is an interesting trapping phenomenon at infinity that limits the amount of decay that can be inferred.

The main theorem for (W)

Consider the conformal wave equation with dissipative bc:

$$\Box_{g_{AdS}} u + 2u = 0 \quad \text{with} \;\; r^2 \partial_r (ru) + \partial_t (ru) = 0 \text{ on } \mathcal{I}. \qquad (7)$$

Define the energy density

$$\varepsilon[u] = \sqrt{1+r^2} \left(\frac{(\partial_t u)^2 + u^2}{1+r^2} + \left[\partial_r \left(\sqrt{1+r^2} u \right) \right]^2 + \left| \nabla\!\!\!/ u \right|^2 \right) .$$

Theorem 6 (HLSW)**.** *Any smooth solution of (7) satisfies*

1. *Uniform Boundedness: For any* $0 < T < \infty$ *we have*

$$\int_{\Sigma_T} \frac{\varepsilon[u]}{\sqrt{1+r^2}} r^2 dr d\omega \lesssim \int_{\Sigma_0} \frac{\varepsilon[u]}{\sqrt{1+r^2}} r^2 dr d\omega ,$$

2. *Degenerate integrated decay without derivative loss:*

$$\int_0^\infty dt \int_{\Sigma_t} \frac{\varepsilon[u]}{1+r^2} r^2 dr d\omega \lesssim \int_{\Sigma_0} \frac{\varepsilon[u]}{\sqrt{1+r^2}} r^2 dr d\omega .$$

3. Non-degenerate integrated decay with derivative loss:

$$\int_0^\infty dt \int_{\Sigma_t} \frac{\varepsilon[u]}{\sqrt{1+r^2}} r^2 dr d\omega \lesssim \int_{\Sigma_0} \frac{\varepsilon\left[u\right] + \varepsilon\left[\partial_t u\right]}{\sqrt{1+r^2}} r^2 dr d\omega \,.$$

Remarks

1. Similar statements hold for higher order energies.

2. As a corollary one obtains

$$\int_{\Sigma_t} \frac{\varepsilon\left[u\right]}{\sqrt{1+r^2}} r^2 dr d\omega \lesssim \frac{1}{1+t} \int_{\Sigma_0} \frac{\varepsilon\left[u\right] + \varepsilon\left[\partial_t u\right]}{\sqrt{1+r^2}} r^2 dr d\omega$$

3. We prove analogues of these statements for (M) and (B).

Any statement of decay has to lose derivatives:

Theorem 7 (HLSW)**.** *There exists no constant* $C > 0$ *such that*

$$\int_0^\infty dt \int_{\Sigma_t} \frac{\varepsilon[u]}{\sqrt{1+r^2}} r^2 dr d\omega \leq C \int_{\Sigma_0} \frac{\varepsilon\left[u\right]}{\sqrt{1+r^2}} r^2 dr d\omega\,.$$

Similarly, there exists no continuous positive function $f : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $f(t) \to 0$ *as* $t \to \infty$ *and*

$$\int_{\Sigma_t} \frac{\varepsilon[u]}{\sqrt{1+r^2}} r^2 dr d\omega \leq f(t) \int_{\Sigma_0} \frac{\varepsilon[u]}{\sqrt{1+r^2}} r^2 dr d\omega$$

This should be compared with a cylinder in Minkowski space where these estimates *can* be proven (the solution decays exponentially).
→ Lions, Chen, Lagnese or HLSW

Theorem 4 is proven using the Gaussian beam approximation.

Compare ...

1. Problem 1: The wave equation $\Box_{g_{ESU}} v - v = 0$ on $\mathbb{R}_t \times \mathbb{S}^3_h$ (with the natural product metric of the Einstein cylinder) where $\mathbb{S}^3_h$ is the (say northern) hemisphere of the 3-sphere S^3 with boundary at $\psi = \frac{\pi}{2}$, where (say optimally) dissipative boundary conditions are imposed.

2. Problem 2: The wave equation $\Box_\eta v - v = 0$ on $\mathbb{R}_t \times \mathbb{B}^3$ (with the flat metric) where $\mathbb{B}^3$ is the unit ball with boundary $\mathbb{S}^2$ where dissipative boundary conditions are imposed.

Maxwell case

In the Maxwell case essentially the same argument goes through replacing

$$\varepsilon[u] = \sqrt{1+r^2}\left(|E|^2 + |H|^2\right)$$

where $E_i = F(e_0, e_i)$ and $H_i = \star_{g_{AdS}} F(e_0, e_i)$ for $i = \bar{r}, 1, 2$ and the dissipative conditions

$$r^2\left(E_A + {\epsilon_A}^B H_B\right) \to 0 \quad \text{at } \mathcal{I}$$

→ energy estimates "by hand", $Q_{\mu\nu}$ tracefree
→ T-estimate controls E_A and H_A on boundary
→ X-estimate gives $E_{\bar{r}}$ and $H_{\bar{r}}$.
→ weighted elliptic estimates for first order system.

Bianchi case

In the Bianchi case we replace

$$\varepsilon\left[u\right]=\left(1+r^{2}\right)^{\frac{3}{2}}\left(\left|E\right|^{2}+\left|H\right|^{2}\right)$$

where $E_{AB}=W\left(e_0,e_A,e_0,e_B\right)$ and $H_{AB}=\star_{g_{AdS}}W\left(e_0,e_A,e_0,e_B\right)$ and the dissipative conditions

$$r^{3}\left(E_{AB}-\frac{1}{2}\delta_{AB}E_{C}{}^{C}+\epsilon_{(A}{}^{C}H_{B)C}\right)\to 0 \quad \text{at } \mathcal{I}$$

$\to$ compare the radiative modes "α, $\underline{\alpha}$"

That these are the correct "optimally dissipative" conditions is non-trivial.

Some difficulties encountered

- Using (divergence-free, symmetric, traceless) Bel-Robinson tensor $Q_{abcd} = W^2 + (\star W)^2$ and vectorfield ∂_t does *not* work.
- boundary term reads

$$\int_{\widetilde{\Sigma}_\infty} \frac{1}{2}\epsilon^{AB} E_{A\bar{r}} H_{B\bar{r}} + \epsilon^{AB} E_{AC} H_B{}^C$$

seems to require additional boundary condition!

- go to a reduced system (insert the constraints into the evolution equation *before* starting to derive estimates)
- prove boundedness and integrated decay *at the same time*

$$Y = \sqrt{3}\partial_t + X$$

Final Remarks and Work in Progress

In view of AdS being conformally flat, the equation (B) is the linearization of the full Bianchi equation

$$(\nabla_g)^a W_{abcd} = 0 \,.$$

Our linear estimates form a key ingredient for a full proof of the non-linear stability of AdS *under dissipative boundary conditions.* Other ingredients are

- weighted elliptic estimates for the second fundamental form
- weighted elliptic estimates for the lapse

This is work in progress (HLSW).

Unique Continuation Properties in aAdS

Question: In what way is the trace of a solution of

$$R_{\mu\nu} = \Lambda g_{\mu\nu}$$

on $\mathcal{I}$ "in correspondence" with the solution in the interior?

We start again with toy problem (Klein-Gordon)

$$\Box_g \psi + \alpha\psi + a^\mu \partial_\mu \psi + V\psi = 0$$

for g being an aAdS spacetime.

Classical Unique Continuation

Does Cauchy data on a boundary hypersurface S determine the solution – if it exists – of a PDE uniquely in a neighbourhood of (one side of) the boundary.

For linear equations, we can rephrase this as

Does zero Cauchy data on a boundary hypersurface S imply that the solution of a PDE must vanish in a neighbourhood of (one side of) the boundary.

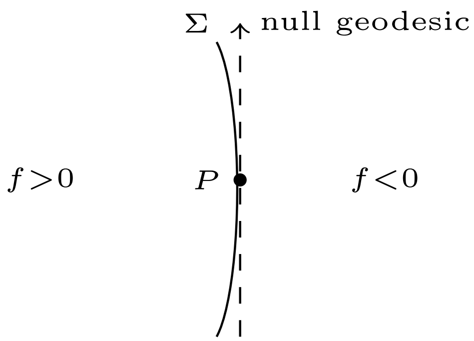
Recall $\Box_\eta \psi = 0$ is *ill-posed* with Cauchy data on a timelike hyperplane (Hadamard)! We only ask for uniqueness here!

Pseudoconvexity

Before discussing AdS, let's consider even simpler case of standard wave equation on Minkowski space, S a timelike hypersurface.

There is a condition on the geometry of S that implies that the unique continuation property holds: **pseudoconvexity**.

Informal (intuitive) Definition: S has a $+$ and a $-$ side. If a null-geodesic tangent to S at a point $p \in S$ remains strictly to the negative side near p, then S is pseudoconvex in the $+$direction.



$\Sigma := \{f = 0\}$ is pseudoconvex (with respect to $\Box_g$ and direction $+$) iff the following holds: $\nabla^2 f\,(X, X) < 0$ on Σ if $g\,(X, X) = Xf = 0$.

Examples

A hyperplane in Minkowski space is not pseudoconvex.
(It is called 0-pseudoconvex as null-geodesics rule the plane S.)

A timelike cylinder in Minkowski space is pseudoconvex in the direction of its interior.

(Alinhac) S not pseudoconvex $\implies$ there exist a and V such that unique continuation fails for $\Box_\eta \psi + a^\mu \partial_\mu \psi + V\psi = 0$!

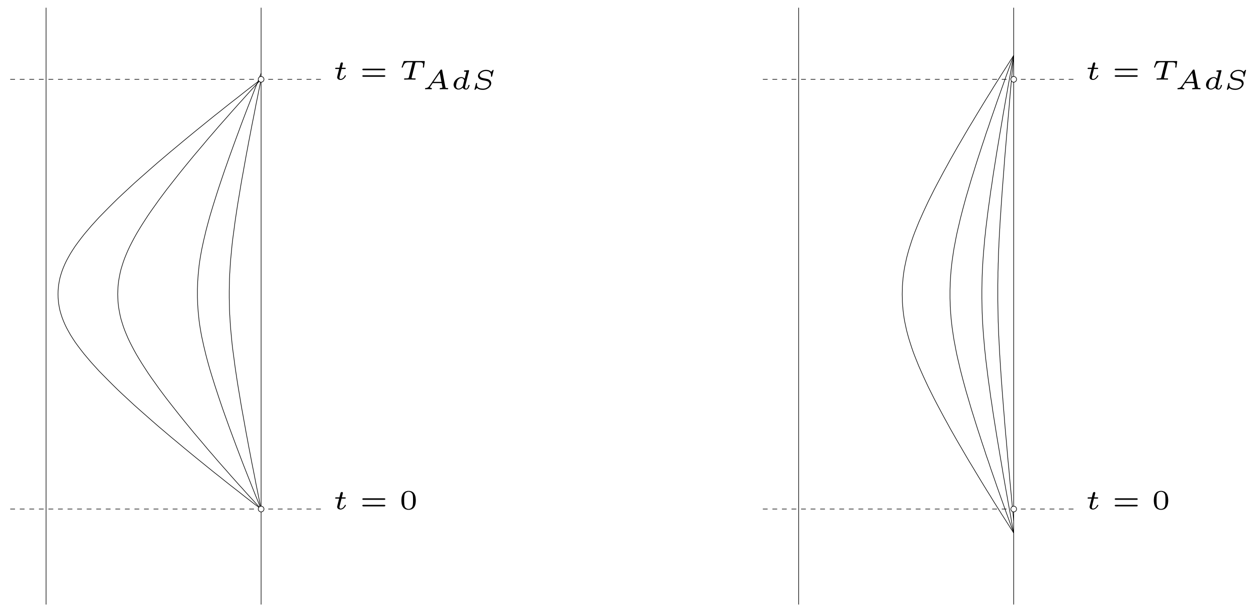
The aAdS case

Cauchy data: Imposing *both* Dirichlet and Neumann conditions for ψ on $\mathcal{I}$ should make the solution unique.

Pseudoconvexity: The boundary $\mathcal{I}$ is 0-pseudoconvex! Level sets of constant r are pseudoconvex (degenerating as $r \to 0$).

Idea: "Bend" the hypersurfaces to construct local foliation of pseudoconvex hypersurfaces.

For pure AdS, length of S_f has to be longer than refocussing time of the null-geodesics!



LHS: Refocussing null-geodesics ($T_{AdS} = \pi$)

RHS: Foliation by level sets of $f(t, r) = \frac{1}{r \sin(ct)}$ with $c < 1$.

Informal statement of the theorem

Theorem 8 (G.H., A. Shao). *Let $\mathcal{M} = (0, R) \times (0, T \cdot \pi) \times S^2$ for $T > 1$ be an asymptotically AdS spacetime patch and ψ satisfy*

$$\Box_g \psi + \alpha\psi + a^\mu \partial_\mu \psi + V\psi = \mathcal{G}\left(\psi, \partial\psi\right)$$

on $\mathcal{M}$ for a and V smooth and suitably decaying near the boundary and suitable conditions on $\mathcal{G}$. Then if both Dirichlet and Neumann conditions hold for ψ on $\mathcal{I}^+$ we have that $\psi \equiv 0$ in a neighborhood of $\mathcal{I}$.

The proof runs via a Carleman estimate exploiting crucially the foliation by (degenerate) pseudoconvex hypersurfaces near $\mathcal{I}$ and an appropriate renormalisation of the wave equation.

More details

Let $\mathcal{M} = (0, R) \times (0, T \cdot \pi) \times S^2$ and equip it with metric

$$g = \frac{d\rho^2 + \mathfrak{g}(\rho)}{\rho^2}$$

where (Fefferman-Graham)

$$\mathfrak{g}(\rho) = \mathring{g}(t, x) - \rho^2 P(t, x) + \rho^3 S(t, x) + ...$$

Note that if g satisfies the vacuum Einstein equations, then P is the Schouten tensor of $\mathring{g}$ and S is the stress-energy tensor of g.

If $\partial_t \mathring{g} = 0$, a good pseudoconvex foliation exists near $\mathcal{I}$ provided

$$P - \frac{1}{T^2} dt^2 + \zeta \mathring{g}$$

is positive definite for some function ζ on $\mathcal{I}$.

We first proved Theorem 1 for $\partial_t \mathring{g} = 0$ (static boundary), CMP 2016.

We are currently writing a paper that obtains the analogous result for $\partial_t \mathring{g} \neq 0$ (length of interval + pc-condition changes)

Main difficulty: "Old" hypersurfaces cease to be pseudoconvex if metric is perturbed in this way. $\rightarrow$ new foliation whose construction involves $\partial_t \mathring{g}$.

The time period required for UC depends on the size of $\partial_t \mathring{g}$.

Generalisations

1. We proved the theorem for a class of tensorial wave equations (applications to the Einstein equations) and higher dimensions

2. Borderline case $T = 1$: If $\mathring{g} = -dt^2 + \gamma_{S^2}$ and g satisfies the Einstein equations, then a uniqueness result can be obtained provided the stress energy tensor S is *negative definite* (work in progress) Applications?

Work in Progress

1. extending Killing symmetries from the boundary to the interior (work in progress)
2. prove a holographic principle for Einstein equations

Summary

1. Well-posedness statements for linear and non-linear toy-models
2. Global estimates for Klein-Gordon on Kerr-AdS
   Dir $\rightarrow$ log-decay, slow!
3. Global estimates for wave, Maxwell and Bianchi on AdS
   fast decay but trapping near the boundary (refocusing)
4. unique continuation results: build on insights from WP theory
   and refocusing phenomenon