Current status of AdS instability

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Anti-de Sitter (AdS) spacetime in $d+1$ dimensions

- AdS is the maximally symmetric solution of the vacuum Einstein equations $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = 0$ with negative Λ (a counterpart of Minkowski ($\Lambda = 0$) and de Sitter ($\Lambda > 0$)):

$$ds^2 = -(1 + r^2/\ell^2) d\tilde{t}^2 + \frac{dr^2}{1 + r^2/\ell^2} + r^2 d\Omega^2_{S^{d-1}}$$

  where $\ell^2 = -d(d-1)/(2\Lambda)$, $r \geq 0$, and $-\infty < t < \infty$.

- Substituting $r = \ell \tan x$ ($0 \leq x < \pi/2$) and $\tilde{t} = \ell\, t$ we get

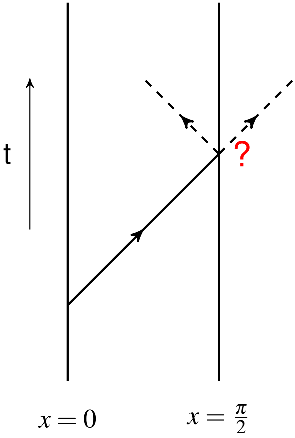
$$ds^2 = \frac{\ell^2}{\cos^2 x} \left(-dt^2 + dx^2 + \sin^2 x \, d\Omega^2_{S^{d-1}} \right)$$

Peculiar causal structure of AdS

$$ds^2 = \frac{\ell^2}{(\cos x)^2}\left[-dt^2 + dx^2 + (\sin x)^2\, d\Omega^2_{S^{d-1}}\right], \; -\infty < t < \infty,\, 0 \leq x < \frac{\pi}{2}$$

Conformal infinity $x = \pi/2$ is the timelike hypersurface $\mathscr{I} = \mathbb{R} \times S^{d-1}$ with the boundary metric $ds^2_{\mathscr{I}} = -dt^2 + d\Omega^2_{S^{d-1}}$

- Null geodesics get to infinity in finite time (but infinite affine length)
- AdS is **not globally hyperbolic** - to make sense of evolution one needs to choose boundary conditions at $\mathscr{I}$
- Asymptotically AdS spacetimes by definition have the same conformal boundary as AdS



Is AdS stable?

- By the positive energy theorem AdS space is the unique ground state among asymptotically AdS spacetimes (much as Minkowski space is the unique ground state among asymptotically flat spacetimes)
- Minkowski spacetime was proved to be asymptotically stable by Christodoulou and Klainerman (1993)
- Key difference between Minkowski and AdS: **the main mechanism of stability of Minkowski - dissipation of energy by dispersion - may be absent in AdS** (for no flux boundary conditions $\mathscr{I}$ acts as a mirror)
- Model for nonlinear dynamics: The problem seems tractable only in $1+1$ dimensions $\Rightarrow$ spherical symmetry $\Rightarrow$ need matter to generate dynamics
  Simple matter model: massless scalar field ϕ in d+1 dimensions

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi G \left(\partial_\alpha \phi \, \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} \partial_\mu \phi \partial^\mu \phi \right)$$

$$g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi = 0, \quad ds^2 = \frac{\ell^2}{\cos^2 x} \left(-A e^{-2\delta} dt^2 + A^{-1} dx^2 + \sin^2 x \, d\Omega^2_{S^{d-1}} \right)$$

Model

- The line element for asymptotically AdS spacetimes at spherical symmetry

$$ds^2 = \frac{\ell^2}{\cos^2 x}\left(-Ae^{-2\delta}dt^2 + A^{-1}dx^2 + \sin^2 x\, d\Omega^2_{S^{d-1}}\right),$$

$(t,x) \in \mathbb{R} \times [0, \pi/2)$.

- Auxiliary variables ($' = \partial_x$, $\dot{} = \partial_t$): $\Pi = A^{-1}e^{\delta}\dot{\phi}$ and $\Phi = \phi'$.
- Field equations (units $8\pi G = d-1$)

$$\delta' = -\frac{\sin 2x}{2}\left(\Phi^2 + \Pi^2\right), \qquad A' = 2(1-A)\frac{d-1-\cos 2x}{\sin 2x} - A\delta',$$

$$\dot{\Pi} = \frac{1}{\tan^{d-1} x}\left(\tan^{d-1} x A e^{-\delta}\Phi\right)', \qquad \dot{\Phi} = \left(Ae^{-\delta}\Pi\right)'.$$

- AdS space: $\phi \equiv 0$, $\delta \equiv \text{const}$, $A \equiv 1$,
- Schwarzschild-AdS: $\phi \equiv 0$, $\delta \equiv \text{const}$, $A \equiv 1 - M\cos^2 x/\tan^{d-2} x$.

Boundary conditions

- Smoothness at the center enforces parity conditions on the fields at $x = 0$ (where Λ is irrelevant)
- Mass function and asymptotic mass:

$$A(t,x) \equiv 1 - m(t,x) \cos^2 x / \tan^{d-2} x$$

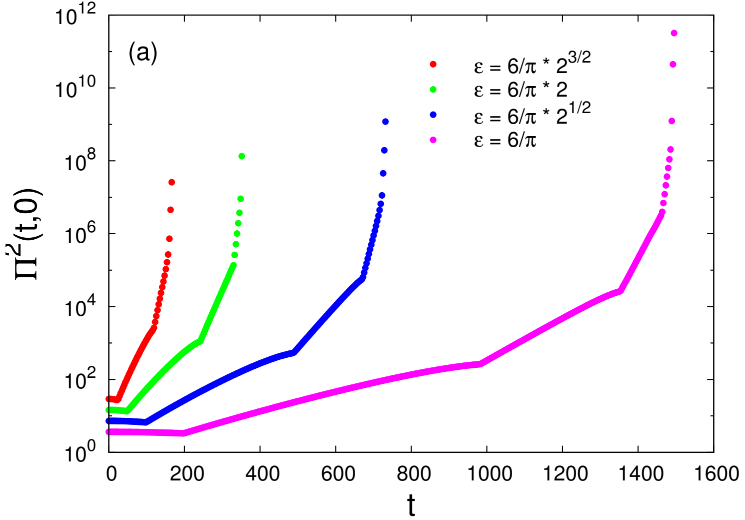
$$M = \lim_{x \to \pi/2} m(t,x) = \int_0^{\pi/2} \left(A\Phi^2 + A\Pi^2\right) (\tan x)^{d-1}\, dx$$

- Smoothness at spatial infinity and the demand for the total mass M to be finite put reflecting boundary conditions on ϕ at $x = \pi/2$,
in particular (using $z = \pi/2 - x$)

$$\phi(t,x) = f_d(t)\, z^d + \mathscr{O}\left(z^{d+2}\right) ,$$
$$A(t,x) = 1 - M z^d + \mathscr{O}\left(z^{d+2}\right) , \quad \delta'(t,x) = \mathscr{O}\left(z^{2d-1}\right) .$$

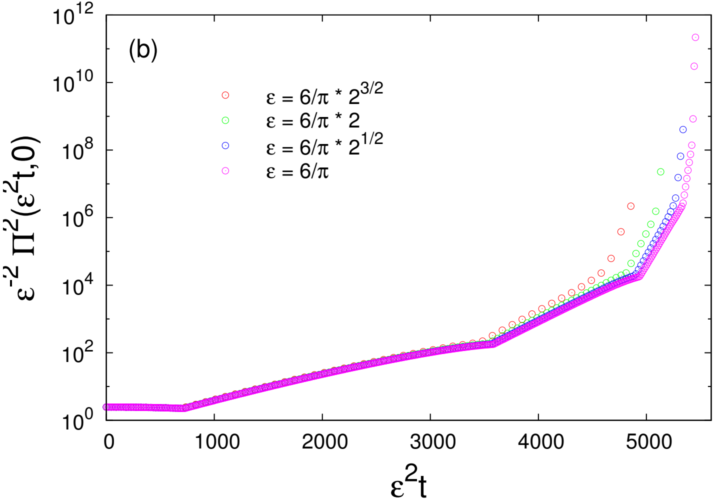
For this model there is no freedom in prescribing boundary conditions
- The problem is locally well-posed [Friedrich, 1995], [Holzegel&Smulevici, 2011] **Animation**

Key numerical evidence for AdS instability



Ricci scalar $R = 2\left(\Phi^2 - \Pi^2\right)/\ell^2 - 12/\ell^2$

Key numerical evidence for AdS instability



Onset of instability at time $t = \mathcal{O}(\varepsilon^{-2})$,

of course in the limit $\varepsilon \to 0$ the argument is based on extrapolation

Conjecture (Bizoń-R. 2011)

AdS$_{d+1}$ (for $d \geq 3$) is unstable under arbitrarily small perturbations (against collapse)

- Numerical evidence: perturbations of size ε collapse in time $\mathcal{O}(\varepsilon^{-2})$.
- The linear spectrum is fully resonant. Nonlinear interactions between harmonics give rise to transfer of energy from low to high frequencies.
- The turbulent cascade leads to concentration of energy on finer and finer spatial scales so eventually a black hole is expected to form.

Spectral properties:

- Linearized equation **[**Ishibashi&Wald, 2004**]**

$$\ddot{\phi} + L\phi = 0\,, \quad L = -\frac{1}{\tan^{d-1} x}\,\partial_x \left(\tan^{d-1} x\, \partial_x\right)\,,$$

- Eigenvalues and eigenvectors of L read ($j = 0, 1, \ldots$):

$L e_j(x) = \omega_j^2 e_j(x)$, with $\omega_j^2 = (d+2j)^2$ and $e_j(x) = N_j \cos^d x\, P_j^{(d/2-1,d/2)}(\cos 2x)\,,$

$$(e_i|e_j) = \int_0^{\pi/2} e_i\, e_j \tan^{d-1} x\, dx = \delta_{ij}$$

$d\omega_j/dj = \pm 2$ the spectrum is nondispersive!

Energy spectrum in $3+1$ dimensions

- Spectral decomposition of the total energy

$$M = \int_0^{\pi/2} \left(A\Phi^2 + A\Pi^2\right) \tan^2 x \, dx = \sum_{j=0}^{\infty} E_j(t)$$

where $E_j := (e_j, \sqrt{A}\,\Pi)^2 + \omega_j^{-2}(e_j', \sqrt{A}\,\Phi)^2$

- Energy spectrum (E_j as a function of j) is an important characteristic of turbulent dynamics
Animation

- Just before collapse $E_j \sim j^{-\alpha}$ with $\alpha \approx 1.2\ (6/5??)$,
[Craps et.al., unpublished**]**, **[**Freivogel&Yang, 2015**]**: in general $\alpha = d - 2$.

Remarks

- Weakly turbulent behavior seems to be common for (non-integrable) nonlinear wave equations on bounded domains (e.g. NLS on torus, [Colliander&Keel, 2008], [Staffilani,Takaoka&Tao, 2008], [Carles&Faou, 2010]) and our work shows that Einstein's equations are not an exception.

- For Einstein's equations the transfer of energy to high frequencies cannot proceed forever because concentration of energy on smaller and smaller scales inevitably leads to the formation of a black hole.

- The role of negative cosmological constant seems to be purely kinematical, that is the only role of Λ is to confine the evolution in an effectively bounded domain. Similar turbulent dynamics has been observed for small perturbations of Minkowski in a box [Maliborski, 2012]

- Generalizations: different matter models (complex scalar field [Buchel,Lehner&Liebling, 2012], Yang-Mills [Maliborski, PhD Thesis 2014]), relaxing symmetry (pure gravity) [Dias,Horowitz&Santos, 2011], instability of AdS_{2+1} [Bizoń&Jałmużna, 2013], Gauss-Bonnet gravity [Deppe et al., 2015], massive scalars [Deppe&Frey, 2015].

"Naive" perturbative expansion

- For small initial data $(\phi, \dot{\phi})_{|t=0} = (\varepsilon f(x), \varepsilon g(x))$:

$$\phi = \varepsilon \phi_1 + \varepsilon^3 \phi_3 + \ldots, \quad \delta = \varepsilon^2 \delta_2 + \varepsilon^4 \delta_4 + \ldots, \quad 1 - A = \varepsilon^2 A_2 + \varepsilon^4 A_4 + \ldots$$

  with $(\phi_1, \dot{\phi}_1)_{|t=0} = (f(x), g(x))$ and $(\phi_j, \dot{\phi}_j)_{|t=0} = (0,0)$ for $j > 1$.

- First order: linearized equation **[Ishibashi&Wald, 2004]** $\ddot{\phi}_1 + L\phi_1 = 0$ gives $\phi_1(t,x) = \sum_{j=0}^{\infty} A_j \cos(\omega_j t + B_j)\, e_j(x)$ with ($L e_j(x) = \omega_j^2 e_j(x), j = 0, 1, \ldots$): $\omega_j^2 = (d + 2j)^2$ and $e_j(x) = N_j \cos^d x\, P_j^{(d/2-1, d/2)}(\cos 2x)$, $L = -\frac{\partial_x(\tan^2 x\, \partial_x)}{\tan^2 x}$
- second order: back reaction on the metric (easy to solve)
- third order: $\ddot{\phi}_3 + L\phi_3 = S$; projection on the basis $\{e_j\}$ gives an infinite set of decoupled forced harmonic oscillations for the generalized Fourier coefficients $c_j(t) := (e_j, \phi_3)$: $\ddot{c}_j + \omega_j^2 c_j = S_j := (e_j, S)$. Then, in general, secular terms arise:

$$\ddot{g}(t) + \omega_0^2\, g(t) = a \cos(\omega t), \quad g(0) = c, \quad \dot{g}(0) = \tilde{c}\,,$$

$$g(t) = \frac{\tilde{c}}{\omega_0} \sin(\omega_0 t) + c \cos(\omega_0 t) + \begin{cases} \dfrac{a\left(\cos(\omega t) - \cos(\omega_0 t)\right)}{\omega_0^2 - \omega^2}, & \omega_0 \neq \omega\,, \\[2ex] \dfrac{a}{2\omega_0}\, t \sin(\omega_0 t)\,, & \omega_0 = \omega\,. \end{cases}$$

Time-periodic asymptotically AdS solutions. Perturbative construction.

- We search for solutions of the form

$$\phi = \varepsilon \cos(\omega_\gamma t) e_\gamma(x)/e_\gamma(0) + \mathcal{O}(\varepsilon^3),$$

  with one *dominant* mode, ε (the $\phi(0,0)$ value) is a small parameter.
- We rescale the time variable

$$\tau = \Omega_\gamma t, \quad \Omega_\gamma = \omega_\gamma + \sum_{\text{even } \lambda \geq 2} \varepsilon^\lambda \omega_{\gamma,\lambda}$$

  and expand the fields perturbatively ε

$$\phi = \varepsilon \cos(\tau) e_\gamma(x) + \sum_{\text{odd } \lambda \geq 3} \varepsilon^\lambda \phi_\lambda(\tau, x),$$

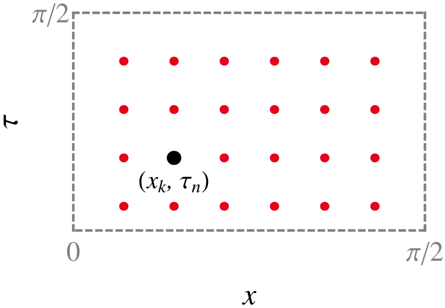
$$\delta = \sum_{\text{even } \lambda \geq 2} \varepsilon^\lambda \delta_\lambda(\tau, x), \qquad 1 - A = \sum_{\text{even } \lambda \geq 2} \varepsilon^\lambda A_\lambda(\tau, x),$$

- We use frequency corrections and integration constants to kill secular terms (order by order) - miraculously it works!

Time-periodic asymptotically AdS solutions. Numerical construction.

$$\phi = \sum_{0\le j<K} f_j(\tau) e_j(x) = \sum_{0\le i<N} \sum_{0\le j<K} f_{i,j} \cos((2i+1)\tau) e_j(x)\,,$$
$$\Pi = \sum_{0\le j<K} p_j(\tau) e_j(x) = \sum_{0\le i<N} \sum_{0\le j<K} p_{i,j} \sin((2i+1)\tau) e_j(x)\,.$$

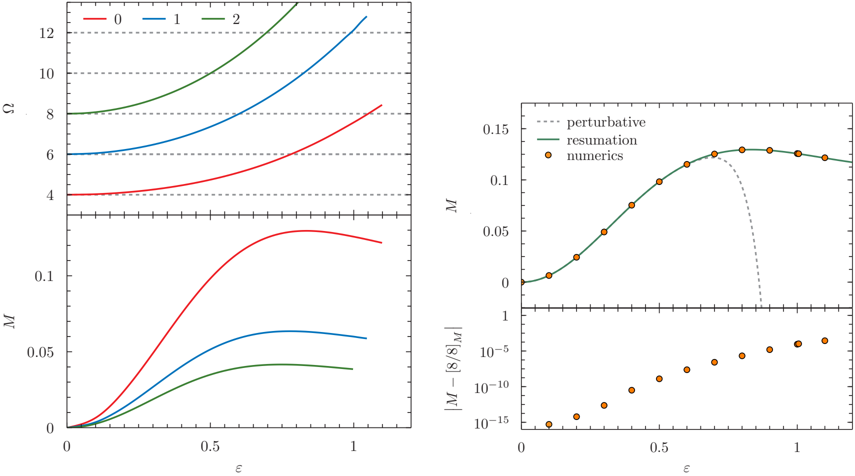
- Find the solution by determining $2\times K\times N+1$ numbers
- Set the equations on a numerical grid of $K\times N$ collocation points
- Add one equation for the normalization condition



$$\sum_{0\le i<N} \sum_{0\le j<K} f_{i,j} e_j(0) = \varepsilon$$

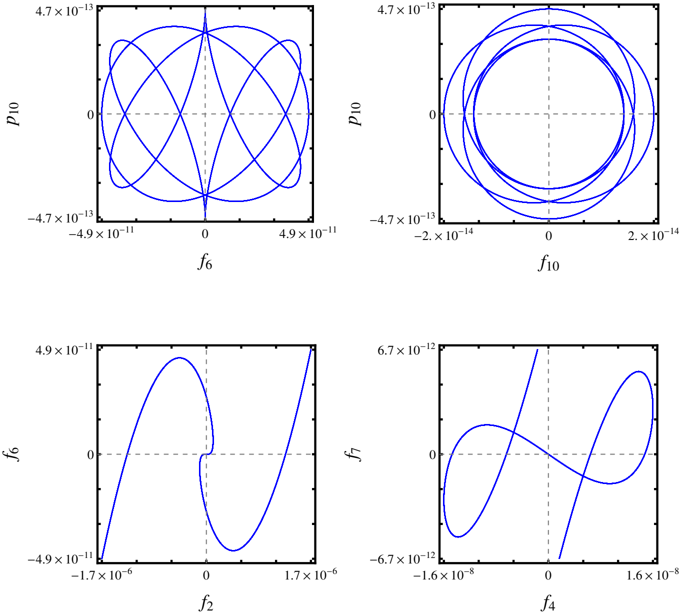
Highly nonlinear system solved with the Newton-Raphson algorithm.

Time-periodic asymptotically AdS solutions (d=4). Results & consistency.



From [Maliborski, PhD Thesis 2014]

Time-periodic solution of Einstein equations [Maliborski,R, 2013]



Closed curves on the slices of phase space – strong **evidence** for the non-linear **stability**. Sections of the phase space spanned by the set of Fourier coefficients

$$\{f_j(\tau), p_k(\tau)\},$$

$$\phi = \sum_j f_j(\tau) e_j(x),$$

$$\Pi = \sum_j p_j(\tau) e_j(x).$$

[Animation (from M. Maliborski)]

Remarks

- There exist (non-linearly) stable time-periodic solutions in Einstein AdS–massless scalar field system
  - further studies and genralization to massive case: [Maliborski, PhD Thesis 2014], [Kim, 2015], [Fodor et al., 2015]
- Cosmological constant confines the evolution in an effectively bounded domain – the possibility of the existence of time-periodic solutions (in contrast to asymptotically flat case)
- Time-periodic solutions in pure vacuum case
  - in the cohomogeneity – two Bianchi IX ansatz ([Bizoń,Chmaj&Schmidt, 2005]): [Maliborski, PhD Thesis 2014]
  - with helical Killing field [Horowitz&Santos, 2014]
- The existence of time-periodic solutions of (non-linear) wave equations on compact domains seems to be common [Maliborski, PhD Thesis 2014]

Resonant approximation for the AdS Einstein-scalar system

- naive perturbative scheme: $\phi = \varepsilon\phi_1 + \varepsilon^3\phi_3 + \ldots$ worked fine for time-periodic solutions, but in general $\phi = \varepsilon\left(\phi_1 + \varepsilon^2 t + \ldots\right)$
- resummed: $\phi_1 = \sum_j A_j(\tau)\cos(\omega_j t + B_j(\tau))e_j(x)$, with "slow time" $\tau = \varepsilon^2 t$ (the time dependence in $A_j(\tau)$, $B_j(\tau)$) used to kill secular terms.

$$2\omega_n \frac{dA_n}{d\tau} = \sum_{\substack{j+k-l=n \\ j\neq n, k\neq n}} S_{jkln} A_j A_k A_l \sin(B_n + B_l - B_j - B_k)$$

$$2\omega_n \frac{dB_n}{d\tau} = T_n A_n^2 + \sum_{j\neq n} R_{jn} A_j^2 + A_n^{-1} \sum_{\substack{j+k-l=n \\ j\neq n, k\neq n}} S_{jkln} A_j A_k A_l \cos(B_n + B_l - B_j - B_k)$$

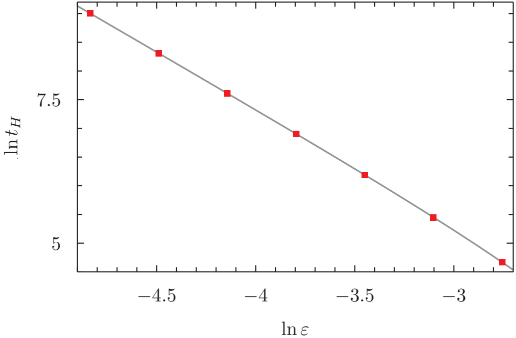
Main advantage: scaling symmetry $B_n(\tau) \to B_n(\varepsilon^2\tau)$, $A_n(\tau) \to \varepsilon A_n(\varepsilon^2\tau) \Longrightarrow$ access to the $\varepsilon \to 0$ limit! Interaction coefficients found numerically in $3+1$ [Balasubramanian et al., 2014], then explicitly [Craps et al., 2014] (for any d)

We have shown that [Bizoń,Maliborski,R., 2015]:

- this infinite system has a solution that becomes singular in finite time
- this singular solution governs the generic blowup

Full GR evolution

- We shall illustrate the numerical results using the time-symmetric two-mode initial data in $4+1$: $\phi(0,x) = \varepsilon \left(\frac{1}{4}e_0(x) + \frac{1}{6}e_1(x)\right)$
- Key observation: horizon forms in time $t_H(\varepsilon) \sim \varepsilon^{-2}$

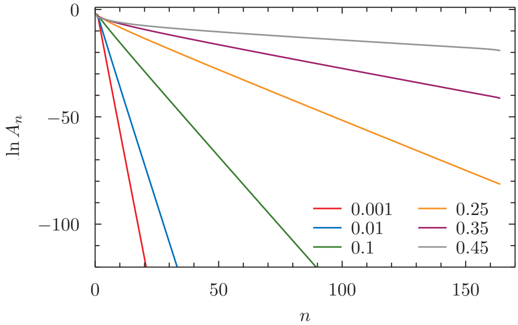


Fitting $t_H = \tau_H \varepsilon^{-2} + C$ we get $\tau_H \approx 0.514$

- This scaling suggests that the instability of AdS should be seen in the resonant approximation.

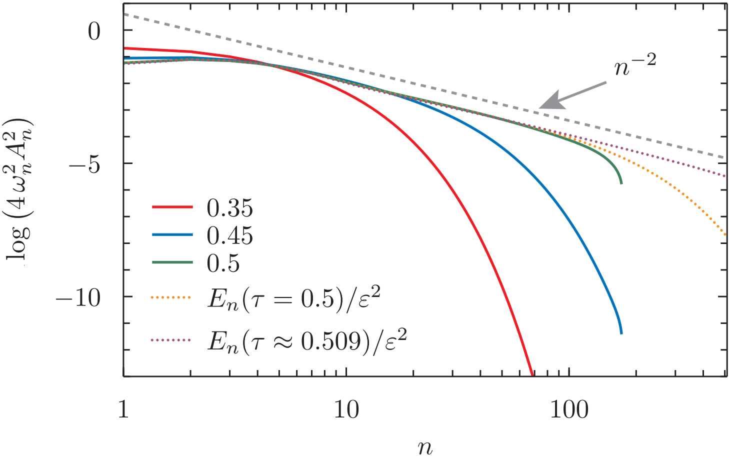
Truncated resonant system - numerics

- For the numerical computation we truncate (RS) at $N = 172$ (TRS)
- For the two-mode initial data the higher modes are quickly excited



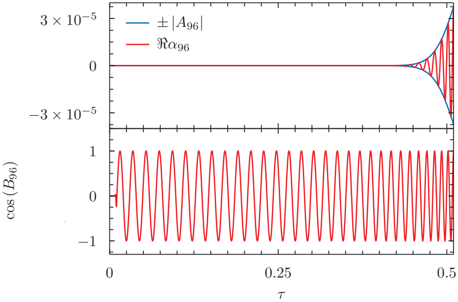
- For early times $A_n(\tau) \sim \tau^{n-1}$ while the phases $B_n(\tau)$ evolve linearly.

Full GR vs. truncated resonant system - energy spectrum



Later times

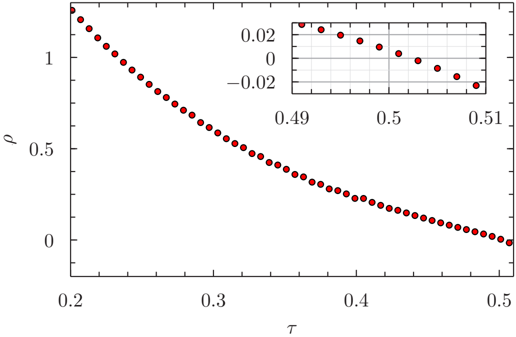
$$\alpha_n = A_n e^{iB_n}$$



- A highly oscillatory behavior develops causing numerical difficulties.
- The time-step of numerical integration, for which the algorithm is convergent, tends to zero as the cutoff N increases.
- This suggests that the solution of (RS) develops an oscillatory singularity in some finite time τ_*.
- Remark: for any finite N the solution of TRS can be numerically continued past τ_*, however this ‘afterlife’ is an artifact of truncation.

Analyticity strip method

- We make the ansatz $A_n(\tau) \sim n^{-\gamma(\tau)} e^{-\rho(\tau) n}$ for large n.
- Fitting to the data we get



- It appears that the 'analyticity radius' $\rho(\tau)$ tends to zero in a finite time τ_*.
- Moreover, the fit reveals that $\lim_{\tau \to \tau_*} \gamma(\tau) = 2$.

Asymptotic analysis

$$2\omega_n \frac{dB_n}{d\tau} = T_n A_n^2 + \sum_{j\neq n} R_{jn} A_j^2 + A_n^{-1} \sum_{\substack{j+k-l=n \\ j\neq n, k\neq n}} S_{jkln} A_j A_k A_l \cos(B_n + B_l - B_j - B_k)$$

- We assume that $A_j(\tau) \sim j^{-2} e^{-\rho_0(\tau_* - \tau)j}$ for large j and $\tau \to \tau_*$
- Asymptotic behavior of the interaction coefficients

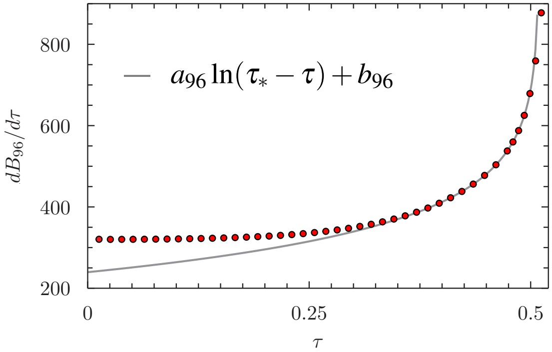
$$T_n \sim n^5, \quad R_{jn} \sim n^2 j^3, \quad S_{\lambda j, \lambda k, \lambda l, \lambda n} \sim \lambda^4 S_{jkln}$$

- The latter implies that $\sum_{\substack{j+k-l=n \\ j\neq n, k\neq n}} S_{jkln}(jkl)^{-2} = \mathcal{O}(1)$
- It follows that for $\tau \to \tau_*$

$$\sum_j R_{jn} A_j^2 \sim n^2 \sum_j j^{-1} e^{-2\rho_0(\tau_* - \tau)j} \sim n^2 \ln(\tau_* - \tau)$$

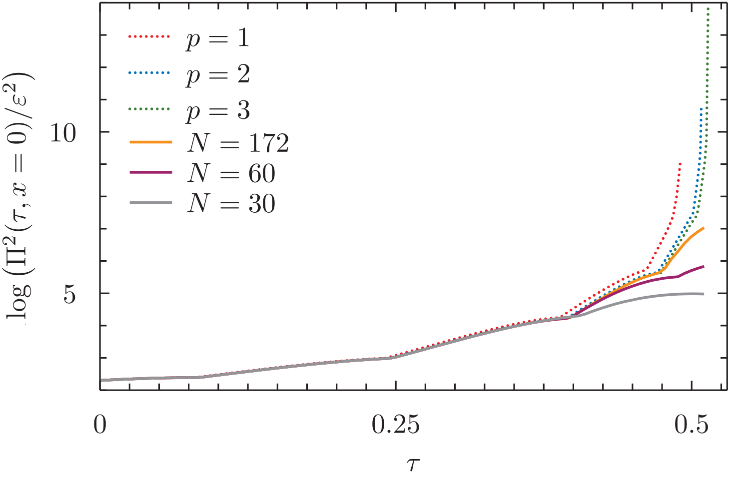
- $\dfrac{dB_n}{d\tau} \sim n \ln(\tau_* - \tau)$ **blows up logarithmically**
- Moreover, B_n behave linearly with n, hence $B_n + B_l - B_j - B_k \approx 0$ for the resonant quartets (confirming that the ansatz is self-consistent).

Numerical confirmation



Performing this fit for all $n > 40$ we confirm that the coefficients a_n and b_n vary linearly with n, while $\tau_* \approx 0.509$ does not depend on n. The blowup time τ_* is close to the collapse time for the true solution $\tau_H := \lim_{\varepsilon \to 0} \varepsilon^{-2} t_H(\varepsilon) \approx 0.514$.

Full GR vs. truncated resonant system - Ricci scalar



$$\varepsilon = (2\pi)^{3/2} 2^{-p}$$

Remarks

- We have constructed the asymptotic solution of the resonant system that becomes singular in finite time.
- Numerics shows that this solution acts as a universal attractor for blowup in the resonant system.
- Key question: how to transfer this blowup result from the resonant system to the full system?
- It is not clear to us what (if any) is the physical interpretation of the oscillatory singularity for the resonant system.
- Nonetheless, the fact that solutions of the resonant system blow up in finite time (for typical initial data) strongly indicates that the corresponding solutions of the full system collapse on the timescale $\mathscr{O}(\varepsilon^{-2})$.

Conclusions

- Dynamics of asymptotically AdS spacetimes is an exceptional meeting point of fundamental problems in general relativity, PDE theory, theory of turbulence, and high energy physics. Understanding of these connections is at its infancy.
- From numerical explorations of Einstein's equations there can grow understanding, conjectures, and roads to proofs and phenomena that would not have been imaginable in the pre-computer era. The role of computation in general relativity seems destined to expand in future.

computer simuations performed at

- *Deszno* supercomputer at Jagiellonian University
- AEI Golm cluster
- PL-Grid, Cyfronet, Kraków

Anti-de Sitter spacetime in $d+1$ dimensions

Geomerically, AdS$_{d+1}$ is wrapped around hyperboloid

$$-\left(T^1\right)^2-\left(T^2\right)^2+\left(X^1\right)^2+\cdots+\left(X^d\right)^2=-\ell^2$$

embedded in flat, $d+2$ dimensional space with line element

$$ds^2=-\left(dT^1\right)^2-\left(dT^2\right)^2+\left(X^1\right)^2+\cdots+\left(X^d\right)^2 .$$

- Global coordinates: $T^1=\ell\sec x\cos t,\quad T^2=\ell\sec x\sin t,\quad X^k=\ell\tan x\, n^k$ with $-\infty<t<+\infty,\quad 0\le x<\pi/2,\quad \sum_{k=1}^d\left(n^k\right)^2=1$; the induced metric:

$$ds^2=\frac{\ell^2}{(\cos x)^2}\left[-dt^2+dx^2+(\sin x)^2\,d\Omega^2_{S^{d-1}}\right]$$

- Poincaré patch:

$$\left.\begin{array}{rl} T^1-X^d & =\ell^2/z>0 \\ T^2 & =\ell t/z \\ X^i & =\ell x^i/z \end{array}\right\}\Rightarrow T^1+X^d=z\left(1+\frac{\vec{x}^2-t^2}{z^2}\right)$$

induced metric: $ds^2=\ell^2\frac{-dt^2+d\vec{x}^2+dz^2}{z^2}$.

Linear stability of AdS

- Scalar wave equation $\Box_g \psi - \mu^2 \psi = 0$ after separation of angular variables $\psi(t,x,\omega) = \sum \phi_A(t,x) S_A(\omega)$ reduces to

$$\ddot{\phi} + L\phi = 0\,, \quad L = -\frac{1}{\tan^{d-1} x}\partial_x \left(\tan^{d-1} x\, \partial_x\right) + \frac{\mu^2}{\cos^2 x} + \frac{l(l+d-2)}{\sin^2 x}$$

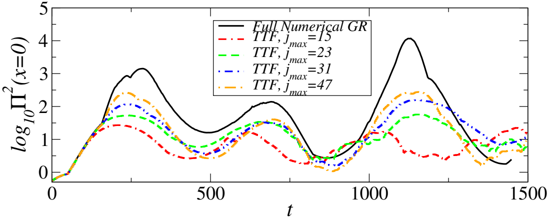
- L is positive for $\mu^2 \geq \mu^2_{BF} = -d^2/4$ and essentially self-adjoint on $L^2([0,\pi/2], \tan^d x\, dx)$ for $\mu^2 \geq \mu^2_{BF} + 1$. For $\mu^2_{BF} \leq \mu^2 < \mu^2_{BF} + 1$ there is a one-parameter freedom of choosing a reflecting boundary condition at $\pi/2$ (Breitenlohner-Freedman 1982, Ishibashi-Wald 2004)
- Electromagnetic and gravitational perturbations are governed by the same operator but with different parameters (d, μ^2)
- Conclusion: for reflecting boundary conditions **AdS is linearly stable** under scalar, electromagnetic, and gravitational perturbations
- Dirichlet eigen-frequencies

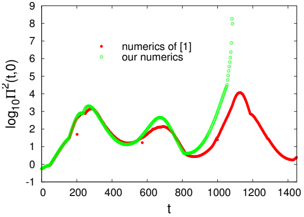
$$\pm\omega_k = 2k + 1 + \frac{1}{2}\sqrt{d^2 + 4\mu^2} + \frac{1}{2}\sqrt{(d-2)^2 + l(l+d-2)}$$

$$\frac{d\omega_k}{dk} = \pm 2 \quad \Rightarrow \quad \text{waves are } \textbf{nondispersive}$$

V. Balasubramanian et al., *Holographic Thermalization, stability of AdS, and the Fermi-Pasta-Ulam-Tsingou paradox*, PRL113, 071601 (2014)

Two-modes initial data and the inverse cascade.





Resonant approximation (example) $\Box_g \phi - \phi^3 = 0$ on AdS$_5$

$$\partial_{tt}\phi + L\phi + \sec^2 x\, \phi^3 = 0, \quad L = -\tan^{-3} x\, \partial_x(\tan^3 x\, \partial_x) \quad (\star)$$

- Linear spectrum: $Le_n = \omega_n^2 e_n$ where $\omega_n^2 = (2n+4)^2$ $(n = 0, 1, \dots)$
- Plugging the mode expansion $\phi(t,x) = \sum_n c_n(t) e_n(x)$ into $(\star)$ we get

$$\frac{d^2 c_n}{dt^2} + \omega_n^2 c_n = \sum_{jkl} I_{jkln}\, c_j c_k c_l, \qquad I_{jkln} = -(e_j e_k e_l \sec^2 x, e_n)$$

- In the interaction picture, defined by variation of constants,

$$c_n = \beta_n e^{i\omega_n t} + \bar{\beta}_n e^{-i\omega_n t}, \quad \frac{dc_n}{dt} = i\omega_n \left(\beta_n e^{i\omega_n t} - \bar{\beta}_n e^{-i\omega_n t}\right)$$

  this becomes

$$2i\omega_n \frac{d\beta_n}{dt} = \sum_{jkl} I_{jkln}\, c_j c_k c_l\, e^{-i\omega_n t}$$

- Each term in the sum has a factor $e^{-i\Omega t}$, where $\Omega = \omega_n \pm \omega_j \pm \omega_k \pm \omega_l$.
  Two kinds of terms: $\Omega = 0$ (**resonant**) and $\Omega \neq 0$ (non-resonant).

Resonant approximation (example) - continued

- We define the slow time $\tau = \varepsilon^2 t$ and rescale $\beta_n(t) = \varepsilon \alpha_n(\tau)$.
- The non-resonant terms $\propto e^{-i\Omega\tau/\varepsilon^2}$ are highly oscillatory for small ε and therefore negligible (at least for some time).
- Keeping only the resonant terms (which is equivalent to time-averaging), we obtain the infinite autonomous dynamical system (**resonant system**)

$$2i\omega_n \frac{d\alpha_n}{d\tau} = \sum_{jkl} I_{jkln}\, \alpha_j \alpha_k \bar{\alpha}_l \,,$$

  where the summation runs over the set of indices $\{jkl\}$ for which $\Omega = 0$ and $I_{jkln} \neq 0$. This set can be shown to reduce to $\{jkl \,|\, j+k-l=n\}$.
- The resonant system is invariant under the scaling $\alpha_n(\tau) \to \varepsilon^{-1}\alpha_n(\tau/\varepsilon^2)$
- The resonant approximation is valid on the timescale $\mathcal{O}(\varepsilon^{-2})$. Thus, on this timescale the dynamics of solutions of the cubic wave equation is dominated by resonant interactions.