Vrije Universiteit Brussel

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AdS (in)stability: an analytic approach and its interplay with numerics

Ben Craps

BC, Oleg Evnin, Joris Vanhoof, JHEP 1410 (2014) 48
BC, Oleg Evnin, Joris Vanhoof, JHEP 1501 (2015) 108
BC, Oleg Evnin, Joris Vanhoof, JHEP 1510 (2015) 79
BC, Oleg Evnin, Puttarak Jai-akson, Joris Vanhoof, JHEP 1510 (2015) 80
BC, Oleg Evnin, Fortsch. Phys. 64 (2016) 336

NumHol2016, Santiago de Compostela, 27 June 2016

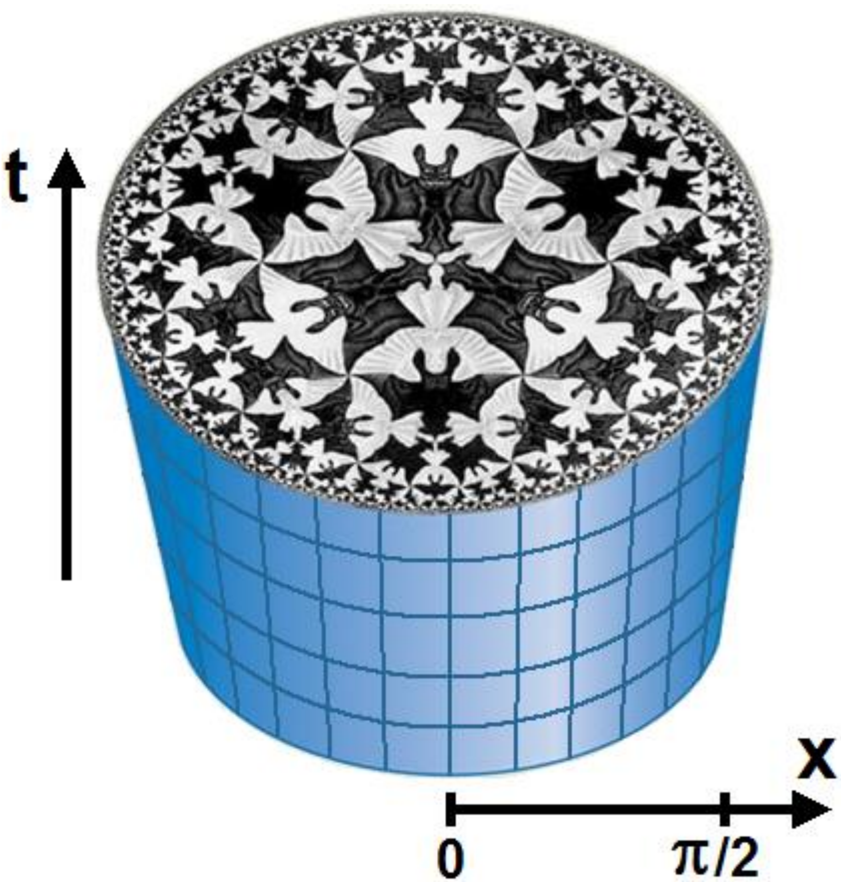
Minkowski space is stable due to dispersion of energy to infinity

Consider, for instance, an infalling spherical shell of massless scalar field minimally coupled to gravity.

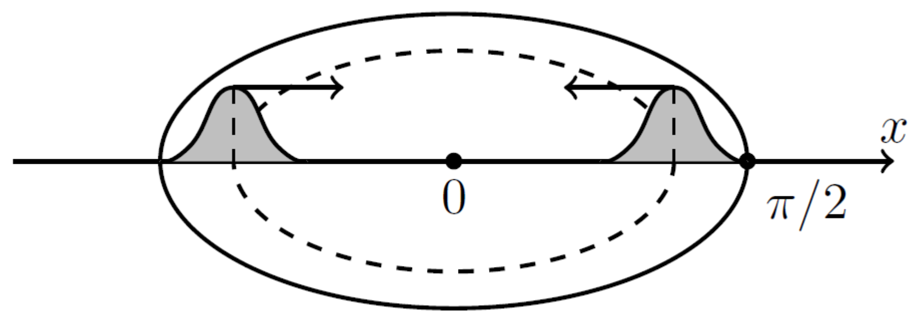
For sufficiently large amplitude: black hole formation.

For sufficiently small amplitude: shell scatters back to infinity. In general, Minkowski space has been shown to be stable under small perturbations. [Christodoulou, Klainerman 1993]

AdS acts like a box: Is it stable?



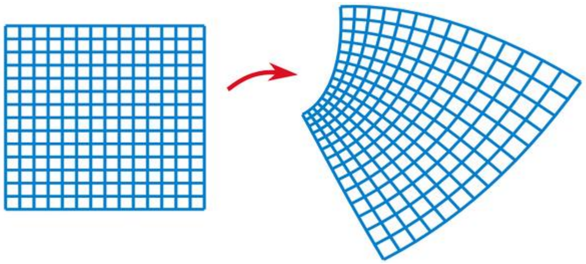
If a shell has too small amplitude to form a black hole right away, it scatters back to the boundary. But the boundary will reflect the shell, which can keep trying to form a black hole, slightly changing its shape every time.



Holography relates black hole formation to thermalization

AdS

CFT



- Anti-de Sitter spacetime
- Black hole
- Black hole formation

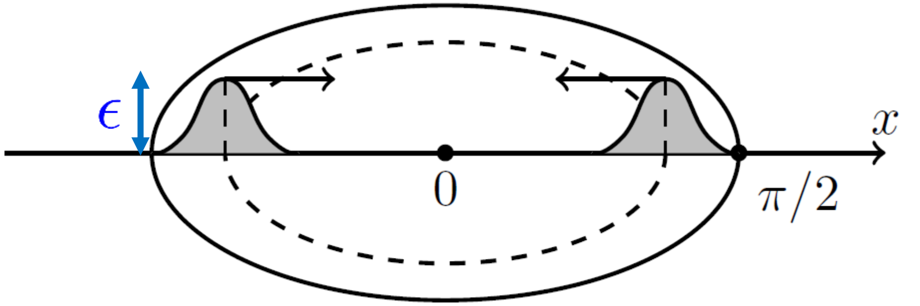
- Conformal Field Theory
- Thermal state
- Thermalization

Conjecture: arbitrarily small perturbations lead to black hole formation in $AdS_{\geq 4}$

For Gaussian shells with amplitude ϵ (and specific width), Bizoń and Rostworowski numerically found BH formation at times scaling like $1/\epsilon^2$, due to turbulent energy flow to short wavelengths (eventually cut off by horizon formation).

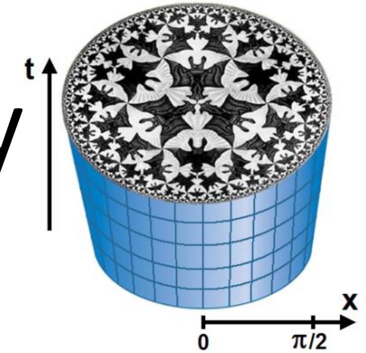
Naive perturbation theory in ϵ shows energy transfer to high frequencies, but breaks down at times of order $1/\epsilon^2$.

For special initial data, with only one normal mode excited, both perturbation theory and numerics indicate regular evolution, at least up to times of order $1/\epsilon^2$.



[Bizoń, Rostworowski 2011]

Study scalar in AdS_{d+1} perturbatively



Spherically symmetric perturbations $\phi = \phi(x,t)$ and

$$ds^2 = \frac{L^2}{\cos^2 x}\left(\frac{dx^2}{A(x,t)} - A(x,t)e^{-2\delta(x,t)}dt^2 + \sin^2 x\, d\Omega^2_{d-1}\right)$$

Metric determined by constraints → Solve e.o.m. for ϕ

Perturbative expansion $\phi = \epsilon\phi_{(1)} + \epsilon^3\phi_{(3)} + \ldots$

Expansion in normal modes $e_n(x)$ with $\omega_n = d + 2n$

$$\phi_{(1)}(x,t) = \sum_{n=0}^{\infty} a_n \cos(\omega_n t + b_n) e_n(x), \quad \phi_{(3)}(x,t) = \sum_{n=0}^{\infty} c_n(t) e_n(x)$$

→ $\ddot{c}_n + \omega_n^2 c_n = \Omega_{ijkn} a_i a_j a_k \cos((\omega_i \pm \omega_j \pm \omega_k)t + (b_i \pm b_j \pm b_k)) + \ldots$

specific complicated integrals involving AdS mode functions

[Bizoń, Rostworowski 2011]

Secular terms suggest energy transfer to UV, but invalidate perturbation theory

$$\phi(x,t) = \sum_{n=0}^{\infty} \left[\epsilon a_n \cos(\omega_n t + b_n) + \epsilon^3 c_n(t) + \ldots\right] e_n(x)$$

$$\ddot{c}_n + \omega_n^2 c_n = \Omega_{ijkn} a_i a_j a_k \cos((\omega_i \pm \omega_j \pm \omega_k)t + (b_i \pm b_j \pm b_k)) + \ldots$$

Resonant if $\pm\omega_n = \omega_i \pm \omega_j \pm \omega_k$

Integer normal mode spectrum $\omega_n = d + 2n$ → many resonances!

Resonances lead to secular terms

$$c_n(t) = \Omega_{ijkn} a_i a_j a_k \, t \sin(\omega_n t + (b_i \pm b_j \pm b_k)) + \ldots$$

They invalidate naive perturbation theory on time scales $t \sim 1/\epsilon^2$.

[Bizoń, Rostworowski 2011]

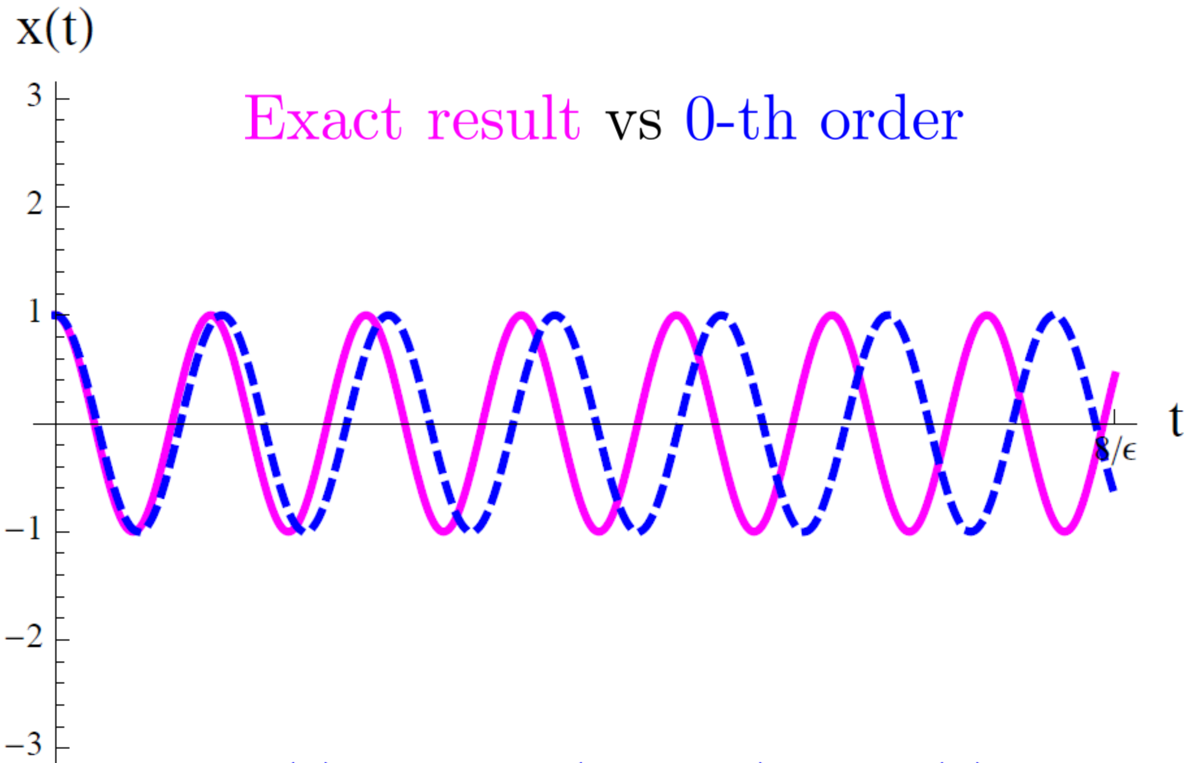
Simpler example: anharmonic oscillator

Particle in potential $V(x) = \frac{\omega^2}{2}x^2 + \frac{\epsilon}{4}x^4$

Equation of motion: $\ddot{x} + \omega^2 x + \epsilon x^3 = 0$

Perturbative expansion: $x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \ldots$

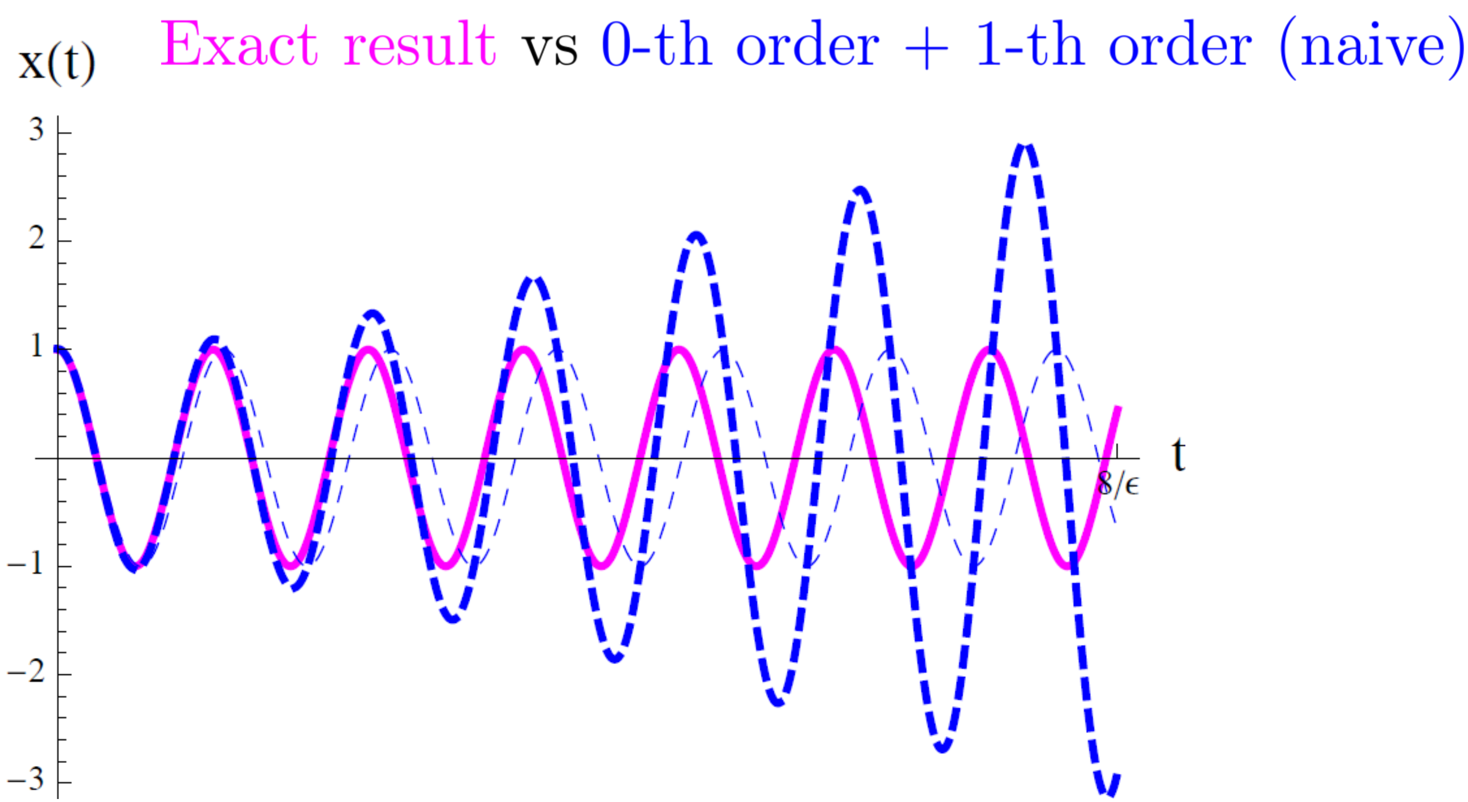
Secular terms invalidate naive perturb. theory for anharmonic oscillator



$$x(t) = a\cos(\omega t + b) + \mathcal{O}(\epsilon)$$

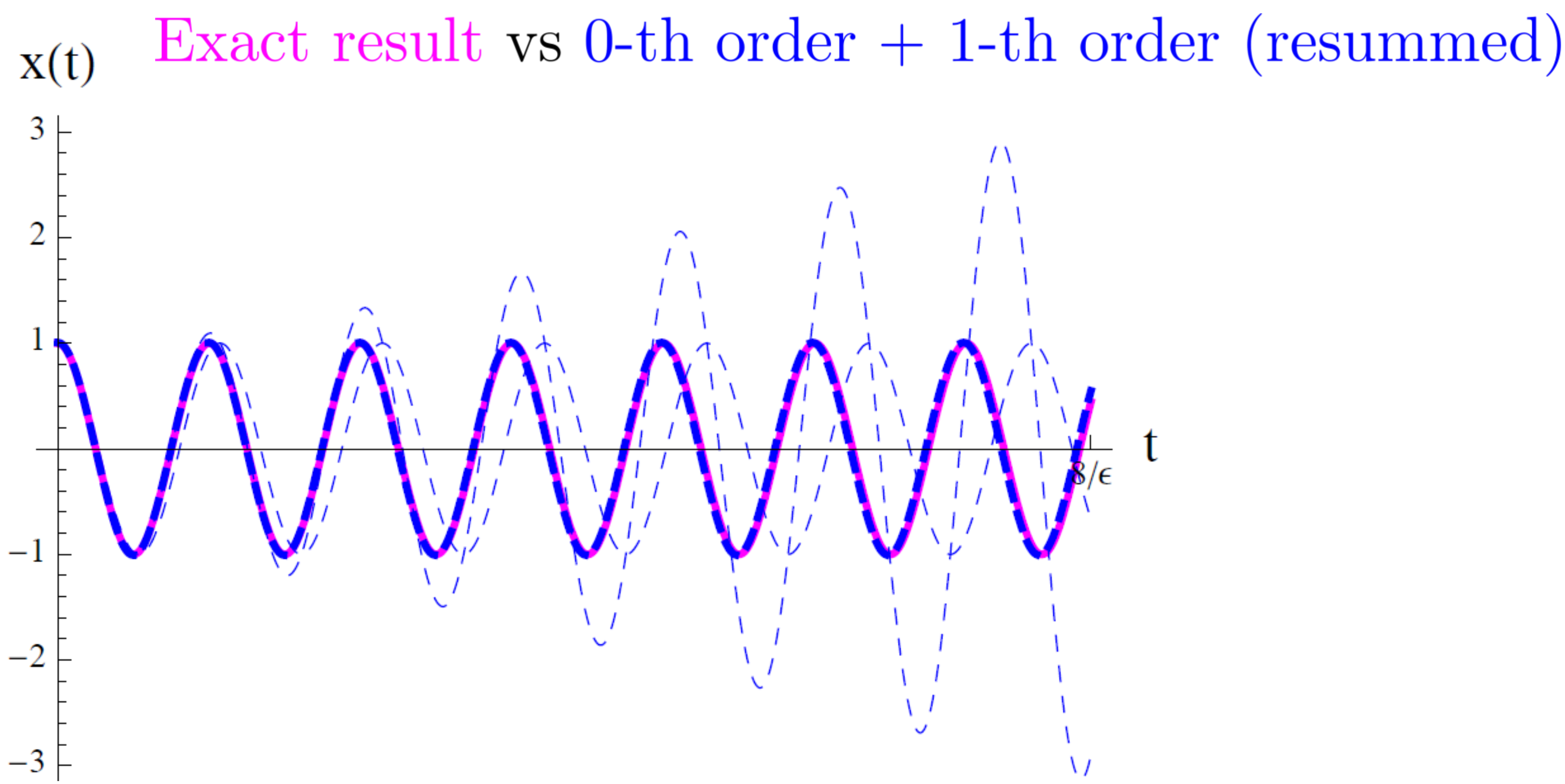
Plot: $\omega = 1,\ \epsilon = 0.2,\ a = 1,\ b = 0$

Secular terms invalidate naive perturb. theory for anharmonic oscillator



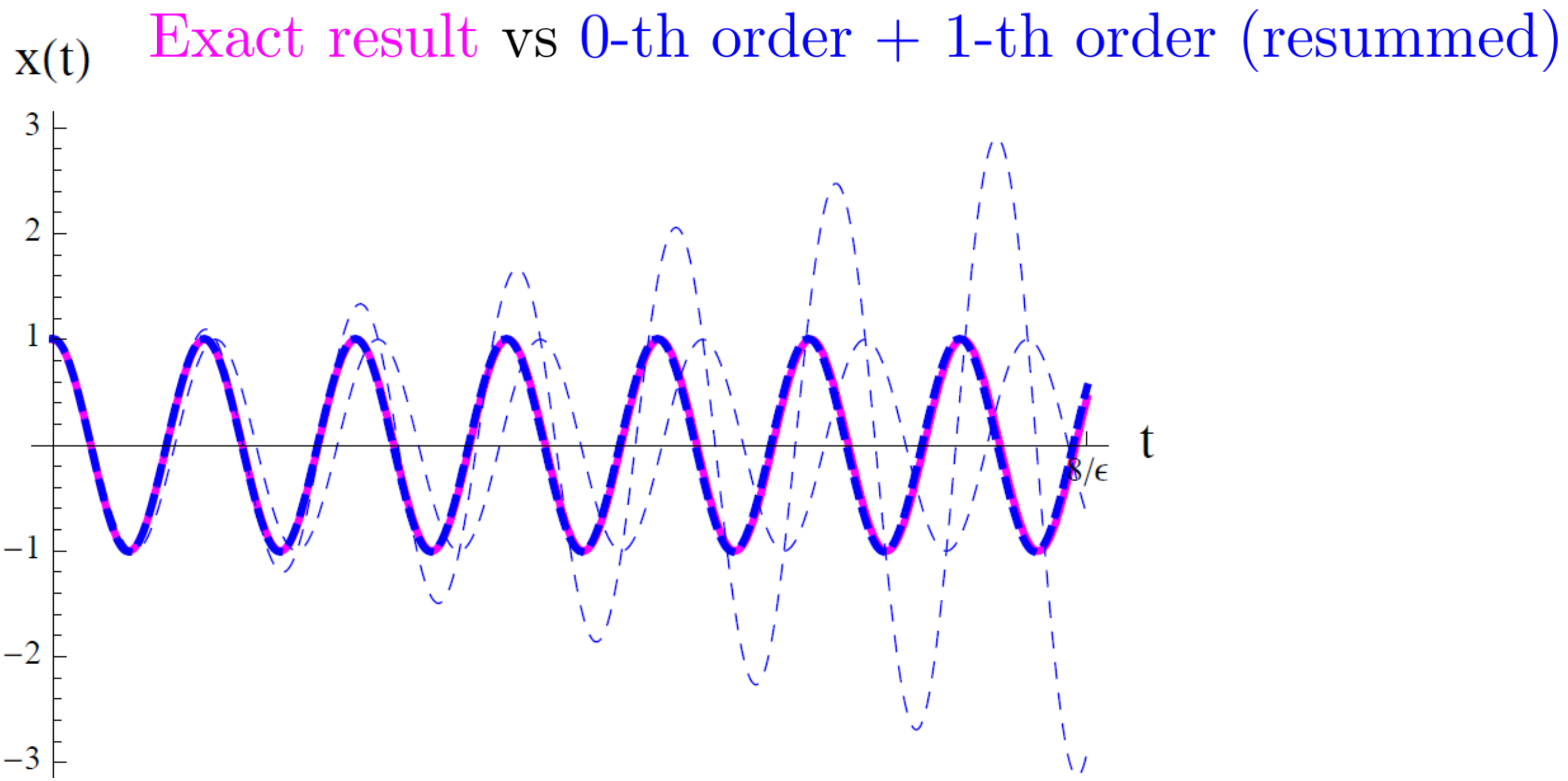
$$x(t) = a\cos(\omega t + b) + \left(\frac{a^3}{32\omega^2}\cos(3\omega t + 3b) - \frac{3a^3}{8\omega}t\sin(\omega t + b)\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

Resummation cures perturbation theory



$$x(t) = a\cos\left(\omega t + b + \frac{3a^2}{8\omega^2}\epsilon t\right) + \frac{a^3}{32\omega^2}\cos\left(3\omega t + 3\left(b + \frac{3a^2}{8\omega^2}\epsilon t\right)\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

The secular term has been resummed into an innocent frequency shift



$$x(t) = a\cos\left(\left(\omega + \frac{3a^2}{8\omega^2}\epsilon\right)t + b\right) + \frac{a^3}{32\omega^2}\cos\left(3\left(\omega + \frac{3a^2}{8\omega^2}\epsilon\right)t + 3b\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

[Poincaré, Lindstedt]

Single-mode initial data: periodic solutions belonging to islands of stability

Using frequency shifts, time-periodic solutions dominated by a single normal mode can be constructed to all orders in ϵ .

[Bizon, Rostworowski 2011]
[Dias, Horowitz, Santos 2011]
[Maliborski, Rostworowski 2013]

Numerical GR indicates that these solutions are stable. They belong to islands of stability.

[Dias, Horowitz, Marolf, Santos 2012]
[Maliborski, Rostworowski 2013]
[Buchel, Liebling, Lehner 2013]
[Maliborski, Rostworowski 2014]

Main messages of this talk

- Numerics and perturbation theory suggest that arbitrarily small scalar perturbations (size ϵ) of AdS can form black holes in times of order $1/\epsilon^2$. Non-collapsing initial data also exist.

- In perturbation theory, the resonant normal mode spectrum of AdS leads to secular terms. Their resummation results in effective flow equations governing the dynamics up to times of order $1/\epsilon^2$ (unless non-linearities become too strong before that time). Peculiarities of AdS enforce a set of selection rules forbidding certain terms in the effective flow equations. These restrictions generate an extra conservation law and have a number of dynamical consequences.

- The short-wavelength asymptotics of this effective dynamics is particularly relevant for understanding the onset of instability. Analytical and numerical progress has recently been made.

Frequency shifts are often not enough

$$\phi(x,t) = \sum_{n=0}^{\infty} \left[\epsilon a_n \cos(\omega_n t + b_n) + \epsilon^3 c_n(t) + \ldots\right] e_n(x)$$

$$\ddot{c}_n + \omega_n^2 c_n = \Omega_{ijkn} a_i a_j a_k \cos((\omega_i \pm \omega_j \pm \omega_k)t + (b_i \pm b_j \pm b_k)) + \ldots$$

Resonances $\pm\omega_n = \omega_i \pm \omega_j \pm \omega_k$ lead to secular terms

$$c_n(t) = \Omega_{ijkn} a_i a_j a_k \, t \sin(\omega_n t + (b_i \pm b_j \pm b_k)) + \ldots$$

$$= (\ldots) t \sin(\omega_n t + b_n) + (\ldots) t \cos(\omega_n t + b_n) + \ldots$$

present for non-trivial resonances;
cannot be absorbed in frequency shifts

[Bizoń, Rostworowski 2011]

Other secular terms can also be resummed

$$x(t) = a\cos(\omega t + b) + (\ldots)\epsilon + (\ldots)\,\epsilon t\sin(\omega t + b)$$
$$+ (\ldots)\,\epsilon t\cos(\omega t + b) + \mathcal{O}\left(\epsilon^2\right)$$

Resummation methods used for AdS (in)stability problem:

- Frequency shifts — [Bizoń, Rostworowski 2011; Dias, Horowitz, Santos 2012; Maliborski, Rostworowski 2013]
- Multiscale analysis — [Balasubramanian, Buchel, Green, Lehner, Liebling 2014]
- Renormalization — [BC, Evnin, Vanhoof 2014]
- Time-averaging — [Basu, Krishnan, Saurabh 2014; BC, Evnin, Vanhoof 2015]

Result: secular terms replaced by slow time dependence:

$$x(t) = a(\epsilon t)\cos(\omega t + b(\epsilon t)) + (\ldots)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

Slow time-dependence is governed by flow equations
$$\begin{cases} \dot{a} = \epsilon(\ldots) \\ \dot{b} = \epsilon(\ldots) \end{cases}$$

Time-averaging leads to flow equations

$$\boxed{\ddot{c}_n + \omega_n^2\, c_n = S_n(c)}$$ ← approximate as cubic

Hamiltonian form: $\dot{c}_n = \pi_n, \qquad \dot{\pi}_n = -\omega_n^2\, c_n + S_n(c)$

Introduce new (complex) variables $\alpha_n(t)$ ("interaction picture")

$$\begin{cases} c_n = \epsilon(\alpha_n e^{-i\omega_n t} + \bar{\alpha}_n e^{i\omega_n t}) \\ \pi_n = -i\epsilon\omega_n(\alpha_n e^{-i\omega_n t} - \bar{\alpha}_n e^{i\omega_n t}) \end{cases} \rightarrow \boxed{\dot{\alpha}_n = \epsilon^2\, S_n(\alpha, \bar{\alpha}, t)}$$

Average S_n over explicit time-dependence: $\boxed{\dot{\alpha}_n = \epsilon^2\, S_n(\alpha, \bar{\alpha})_{\text{av}}}$

("integrate out fast oscillations") "flow equations"

Flow equations are reliable on time interval of order $1/\epsilon^2$

(unless amplitude growth invalidates cubic approximation)

Averaging is equivalent to multiscale and RG at this order.

[BC, Evnin, Vanhoof 2015]

Secular terms are replaced by flow of amplitudes and phases

$$\phi(x,t) = \sum_{n=0}^{\infty} \left[\epsilon a_n \cos(\omega_n t + b_n) + \epsilon^3 c_n(t) + \ldots\right] e_n(x)$$

$$\begin{aligned} c_n(t) = & \ \Omega_{ijkn} a_i a_j a_k \, t \sin(\omega_n t + (b_i \pm b_j \pm b_k)) + \ldots \\ = & \ (\ldots) t \sin(\omega_n t + b_n) + (\ldots) t \cos(\omega_n t + b_n) + \ldots \end{aligned}$$

$$\phi(x,t) = \sum_{n=0}^{\infty} \left[\epsilon \, a_n(\epsilon^2 t) \cos(\omega_n t + b_n(\epsilon^2 t)) + \ldots\right] e_n(x)$$

Flow equations: $\begin{cases} \dot{a}_n = \epsilon^2(\ldots) \\ \dot{b}_n = \epsilon^2(\ldots) \end{cases}$ $\left(\alpha_n = \frac{a_n}{2} e^{-ib_n}\right)$

Many flow channels are closed

$$\phi(x,t) = \sum_{n=0}^{\infty} \left[\epsilon a_n \cos(\omega_n t + b_n) + \epsilon^3 c_n(t) + \ldots\right] e_n(x)$$

$$\ddot{c}_n + \omega_n^2 c_n = \Omega_{ijkn} a_i a_j a_k \cos((\omega_i \pm \omega_j \pm \omega_k)t + (b_i \pm b_j \pm b_k)) + \ldots$$

Normal mode spectrum $\omega_n = d + 2n$ → resonances if

- ~~$\omega_i + \omega_j + \omega_k = \omega_n$~~ (dynamically forbidden)
- $\boxed{\omega_i + \omega_j - \omega_k = \omega_n}$
- ~~$\omega_i - \omega_j - \omega_k = \omega_n$~~ (dynamically forbidden)
- ~~$-\omega_i - \omega_j - \omega_k = \omega_n$~~ (positivity)

[BC, Evnin, Vanhoof 2014] (for lowest modes implicit in earlier work)

Flow equations conserve three charges

In terms of $\alpha_n = \frac{a_n}{2} e^{-ib_n}$, flow equations: $\frac{d\alpha_n}{d\tau} = \frac{i\epsilon^2}{\omega_n} \frac{\partial W}{\partial \bar{\alpha}_n}$, with

$$W = \frac{1}{4} \sum_{\substack{i,j,k,l \\ \omega_i + \omega_j = \omega_k + \omega_l}} \Omega_{ijkl} \alpha_i \alpha_j \bar{\alpha}_k \bar{\alpha}_l$$

Can be obtained from $L = \sum_n i\omega_n \left(\bar{\alpha}_n \frac{d\alpha_n}{d\tau} - \alpha_n \frac{d\bar{\alpha}_n}{d\tau} \right) + 2\epsilon^2 W$

(related to original scalar field Lagrangian by averaging)

Symmetries of averaged Lagrangian lead to 3 conserved charges

$\alpha_n \mapsto e^{i\omega_n \theta} \alpha_n$ → $E = \sum_n \omega_n^2 |\alpha_n|^2$ — conservation of E was previously observed numerically by [Balasubramanian, Buchel, Green, Lehner, Liebling 2014]

$\alpha_n \mapsto e^{i\theta} \alpha_n$ → $J = \sum_n \omega_n |\alpha_n|^2$ — (closed flow channels crucial!)

$\tau \mapsto \tau + \tau_0$ → W — (quartic “interaction energy”)

[BC, Evnin, Vanhoof 2015]; cf. [Basu, Krishnan, Saurabh 2014] for probe scalar field

Conservation laws restrict the dynamics

Families of quasiperiodic solutions with constant amplitudes $|\alpha_n|$ have been found numerically by truncating to a finite number of mode functions. They are thought to anchor islands of stability in the phase space of AdS perturbations.

[Balasubramanian, Buchel, Green, Lehner, Liebling 2014]
[Buchel, Green, Lehner, Liebling 2015]

The abundance of quasiperiodic solutions is intimately related to the missing secular terms in perturbation theory (and therefore to conserved quantities). [BC, Evnin, Vanhoof 2014, 2015]

Conservation of E and J implies dual cascades.

[Buchel, Green, Lehner, Liebling 2015]

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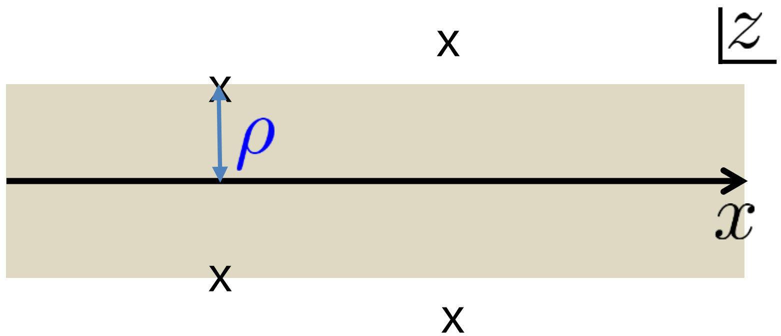
- In perturbation theory, the resonant normal mode spectrum of AdS leads to secular terms. Their resummation results in effective flow equations governing the dynamics up to times of order $1/\epsilon^2$ (unless non-linearities become too strong before that time). Peculiarities of AdS enforce a set of selection rules forbidding certain terms in the effective flow equations. These restrictions generate an extra conservation law and have a number of dynamical consequences.

- The short-wavelength asymptotics of this effective dynamics is particularly relevant for understanding the onset of instability. Analytical and numerical progress has recently been made.

Analyticity strip method: Fourier asymptotics diagnose singularity formation

Consider solution $u(t,x)$ of evolution equation for real-analytic initial data. Analytic extension $u(t,z)$ to complex plane of spatial variable will typically have complex singularities moving in time. If a complex singularity hits the real axis, $u(t,x)$ becomes singular.



Complex singularity closest to real axis: $z = x + i\rho$. Singularity if $\rho(t)$ vanishes in finite time. (ρ determines width of analyticity strip around real axis.)

$\rho(t)$ is encoded in $\exp(-\rho k)$ decay of Fourier coefficients of $u(t,x)$ for large k; it can be obtained from the asymptotics of the Fourier spectrum.

[Sulem, Sulem, Frisch 1983], as described in [Bizoń, Jałmużna 2013]

AdS_3 is unstable, but not to BH formation

Energy threshold between AdS_3 and lightest black hole
→ small perturbations cannot form black hole.

Numerical evidence that analyticity strip shrinks exponentially, does not vanish in finite time → solution is non-singular (no naked singularity).

Nevertheless, AdS_3 is unstable because of turbulent energy flow to short wavelengths. “Small” perturbations do no remain “small”. Technically, higher Sobolev norms (sums of mode energies with short-wavelength contributions preferentially weighted) grow exponentially fast.

[Bizoń, Jałmużna 2013]

Flow equations see onset of collapse

Extrapolating numerical GR to arbitrarily small amplitude ϵ is tricky. In [Bizoń, Rostworowski 2011], $1/\epsilon^2$ scaling of the collapse time was observed, but does the scaling persist to arbitrarily small ϵ ?
cf. [Dimitrakopoulos, Freivogel, Lippert, Yang 2015; Dimitrakopoulos, Yang 2015]

The effective flow equations have $1/\epsilon^2$ scaling built in. If collapse is captured by these equations, extrapolation becomes possible.

Numerical study of flow equations (truncated to 172 normal modes) for collapsing initial data in AdS_5 suggests (for the infinite system) a finite-time oscillatory singularity.

Fit to ansatz $a_n(\epsilon^2 t) \sim n^{-\gamma(\epsilon^2 t)} e^{-\rho(\epsilon^2 t) n} \quad (n \gg 1)$, motivated by analyticity strip method, shows indeed that ρ tends to zero in finite time.

[Bizoń, Maliborski, Rostworowski 2015]; see also [Green, Maillard, Lehner, Liebling 2015] [Deppe 2016]

Analytical methods complement numerics

Using numerically derived UV asymptotics of the interaction coefficients in the flow equations, [Bizoń, Maliborski, Rostworowski 2015] found that their amplitude spectrum seemed consistent with the flow equations.

The UV asymptotics of the interaction coefficients have now been derived analytically for arbitrary dimension. [BC, Evnin, Vanhoof 2015]

The bottleneck in numerical studies of the flow equations was the evaluation of interaction coefficients up to high mode number. A new recursive method was developed to speed this up. [BC, Evnin, Vanhoof 2015]

General qualitative features of quasiperiodic solutions have been explained analytically. [BC, Evnin, Jai-akson, Vanhoof 2015]

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Implications for thermalization in finite volume?

- entanglement entropy oscillations, revivals of the initial state
  [Abajo-Arrastia, da Silva, Lopez, Mas, Serantes 2014]
  [da Silva, Lopez, Mas, Serantes 2014]
- pre-thermalization: small BH as intermediate state
  [Dimitrakopoulos, Freivogel, Lippert, Yang 2015]
- cf. thermalization (or not!) in infinite-volume hard wall model
  [BC, Kiritsis, Rosen, Taliotis, Vanhoof, Zhang 2014;
  BC, Lindgren, Taliotis, Vanhoof, Zhang 2014]
- Is lack of thermalization a large N artifact?
  [Dias, Horowitz, Marolf, Santos 2012]
- Role of additional bulk dimensions in AdS/CFT? Localization instability? [Buchel, Buchel 2015]

Other developments and open questions

- Massive scalar fields coupled to gravity.
  [Kim 2014] [Okawa, Lopes, Cardoso 2015] [Deppe, Frey 2015]
- Scalar fields in Gauss-Bonnet gravity.
  [Deppe, Kolly, Frey, Kunstatter 2015]
- Pure gravity in 4+1d.
  [Bizoń, Rostworowski] (talk at Strings 2014)
- Spherical cavity in Minkowski space. [Maliborski 2012]
  [Maliborski, Rostworowski 2014] [Okawa, Cardoso, Pani 2014]
- Self-interacting probe scalar field. [Basu, Krishnan, Saurabh 2014]
  [Yang 2015] [BC, Evnin, Jai-akson, Vanhoof 2015]
- Stability of spacetimes with asymptotically resonant spectra.
  [Dias, Horowitz, Marolf, Santos 2012] [Menos, Suneeta 2015]

Other developments and open questions

- Are all stability islands anchored on quasiperiodic solutions?
  [Deppe, Frey 2015]
- Fate of quasiperiodic solutions/stability islands for $t \gg 1/\epsilon^2$?
- Can one prove collapse for arbitrarily small initial data?
  [Bizoń, Maliborski, Rostworowski 2015]
- Deeper reason for closed flow channels?
  [Evnin, Krishnan 2015; Evnin, Nivesvivat 2016]
- Beyond spherical symmetry? [Dias, Horowitz, Santos 2012] [Dias, Santos 2016]
- Analytic handle on the turbulent regime?
  cf. [de Oliveira, Pando-Zayas, Rodrigues 2013] for numerical results and [Freivogel, Yang 2016] for a conjectured relation to stationary solutions

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