Heavy quark impurities, holography and entanglement

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Based on work with: Dorian Silvani (Swansea) (1611.06033 and 1711.01554).

Introduction and Motivation

- Defects and impurities play a central role in various physical systems, e.g. Kondo effect
- For CFTs in 2-d with a boundary; with flow triggered on boundary (and critical bulk), the g-theorem applies: there exists a quantity g (boundary entropy) decreasing monotonically under the flow. [Affleck-Ludwig; Friedan-Konnechny]

  g-function: contribution to the Entanglement Entropy (EE) of interval containing the defect.
- Heavy quark "impurities" in gauge theories compute Wilson/Polyakov loops. For generic representations these are computed holographically by objects with nontrivial dynamics.

  Is there a version of the g-theorem via EE for these ?

  What is the complete characterization of Polyakov loops at strong coupling?

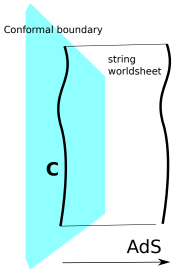
Wilson lines/heavy quarks

- Wilson loops, basic gauge-invariant Yang-Mills observables:

$$W[C] = \frac{1}{\dim[R]} \mathrm{Tr}_R \mathcal{P} \exp\left(i \oint_C A_\mu \, \dot{x}^\mu \, ds \right)$$

Representation "R" and contour C.

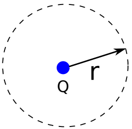
- In large-N theories with holographic gravity/string duals, they are computed by (multiple) open string world sheets [Maldacena (2001); Rey-Theisen-Yee (2001)]



Wilson lines/heavy quarks

- Wilson line $\leftrightarrow$ Phase associated to heavy quark worldline.
- Heavy quark "impurity" interacts with the "ambient" Yang-Mills degrees of freedom:

$$\mathcal{L} = \mathcal{L}_{\rm imp} + \mathcal{L}_{\rm int} + \mathcal{L}_{\rm YM}$$



- Probe this with:

  (i) $\langle \mathcal{O}_{\rm YM} \rangle (r)$: Gauge-invariant Yang-Mills operators

  (ii) $S_{\rm EE}(r)$: Entanglement Entropy across sphere

  (iii) Deformations of $\mathcal{L}_{\rm imp}$

Holographic Wilson lines

- Holographic gauge/string duality for large-N theories: Natural to consider BPS Wilson lines, e.g. in N=4 SUSY Yang-Mills:

$$W[C] = \frac{1}{\mathrm{dim}[R]} \mathrm{Tr}_R \mathcal{P} \exp \left[i \oint_C \left(A_\mu \dot{x}^\mu + \Phi_I n^I(s) |\dot{x}| \right) ds \right]$$

$\{\Phi_I\}_{I=1,\ldots 6}$: Scalars of $\mathcal{N} = 4$ theory. n^I: Unit vector in $\mathbb{R}^6$

(Note: The Euclidean continued thermal/Polyakov line is not unitary)

- Such Wilson lines computed by 2d string worldsheets

or

Higher dimensional wrapped branes with world-volume $\simeq \mathbb{R}^2 \times \mathcal{M}$ for some compact $\mathcal{M}$.

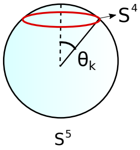
Large rank representations

- When $\dim[R] = k$ with $\frac{k}{N} =$ fixed as $N \to \infty$

  k strings $\longrightarrow$ wrapped D-brane.

  $SU(N)$ $\mathcal{N} = 4$ SUSY Yang-Mills:

- Rank k antisymmetric tensor $(\mathcal{A}_k)$: D5-brane wrapping $S^4 \subset AdS_5 \times S^5$



[Camino-Paredes-Ramallo (2001); Hartnoll-SPK (2006); Yamaguchi (2006)]

- Rank k symmetric tensor $(\mathcal{S}_k)$: D3-brane wraps $S^2 \subset AdS_5$ [Drukker-Fiol (2005)]

Yang-Mills VEVs induced by heavy quarks

- Static quark sources Yang-Mills fields: $\langle \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}\rangle$, $\langle T_{\mu\nu}\rangle \ldots$

- VEVs $\leftrightarrow$ Normalizable modes of AdS_5 fields

- $S_{\rm bulk} \sim N^2 S_{sugra}[\phi, g_{\mu\nu} \ldots] + \int J_{source}$

  't Hooft coupling: $\lambda = g^2_{YM} N \gg 1$

  $J_{\rm F1} \sim \sqrt{\lambda}$ $\qquad$ $J_{\rm D3,D5} \sim N\sqrt{\lambda}$

- Dilaton ϕ $\leftrightarrow$ Scalar "glueball" $\sim \frac{1}{N}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}$

  Metric $g_{\mu\nu}$ $\leftrightarrow$ Stress tensor $T_{\mu\nu}$

Yang-Mills VEVs

E.g: Extracting $\langle\mathcal{O}\rangle = \langle \frac{1}{N} \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}\rangle$:

- Solve linearized e.o.m. of dilaton in presence of source in AdS_5

$$N^2\phi(\vec{x}, z) = \int d^5x' \, G_{\mathrm{AdS}}(x^\mu, z; x'^\mu, z')\, J(x', z')$$

[Danielsson, Kesko-Vakkuri,Kruczenski ('98)]

- Expand RHS near AdS-boundary $z \to 0$ and find coefficient of z^4, given $\Delta_{\mathcal{O}} = 4$.

Conformal impurity with representation $\mathcal{A}_k$

- The straight (BPS) Wilson line in $\mathcal{N} = 4$ SYM has $\langle W \rangle = 1$ for any representation.

- For representation $\mathcal{A}_k$: D5-brane embedding $\simeq AdS_2 \times S^4$

$$\underline{AdS_5}: \quad ds^2 = \frac{dz^2}{z^2} + \frac{1}{z^2}\left(-dt^2 + d\vec{x}^2\right)$$

$$S^5: \quad ds^2 = d\theta^2 + \sin^2\theta \, d\Omega_4^2$$

D5-brane: (t, z, Ω_4) $\quad \vec{x} = 0$

$$\theta = \theta(z)$$

D5-brane embedding

$$S_{\rm D5} = N\frac{\sqrt{\lambda}}{8\pi^2}\left[\int dt\, dr\, d\Omega_4 \sqrt{*g + 2\pi\alpha' F} - \int 2\pi\alpha' F \wedge *C_4\right]$$

$$\frac{\delta S}{\delta F} = -k$$

- World-volume electric field fixes number of units of string charge.
- E.o.m. yields a constant solution: $\theta(z) = \theta_k$

  $$\frac{\pi k}{N} = \theta_k - \sin\theta_k \cos\theta_k$$

- World-volume $\simeq AdS_2 \times S^4 \implies$ Conformal quantum mechanics on impurity

Conformal impurity with representation $\mathcal{A}_k$

- Lightest (linearized) fluctuation of D5-brane embedding: $\delta\theta(z)$

$$m^2 = 12\,, \qquad \Delta = \tfrac{1}{2} + \sqrt{\tfrac{1}{4} + m^2} = 4$$

Irrelevant operator in CFT_1

- For conformal impurity $\langle \mathrm{Tr} F^2 \rangle \sim r^{-4}$ where $(r = |\vec{x}|)$:

$$\frac{1}{N}\langle \mathrm{Tr} F^2 \rangle_{\mathcal{A}_k} = \frac{\sqrt{2}}{24\pi^2}\,\sqrt{\lambda}\,\sin^3\theta_k\,\frac{1}{r^4}$$

- Same factor appears as action for circular Wilson loop and Polyakov loop in rep. $\mathcal{A}_k$

"Collapsed" solution

- The wrapped D5-brane permits a distinct finite action solution with $\theta = \pi$.
- Also conformal with AdS_2 factor and action of k strings or k fundamental quarks
- Lightest fluctuation $\delta\theta(z)$: $m^2 = 0 \leftrightarrow \Delta = 1$ in CFT_1
- $$\frac{1}{N}\langle \mathrm{Tr} F^2 \rangle_{\rm collapsed} = \frac{\sqrt{2}}{24\pi^2}\sqrt{\lambda}\,\frac{3\pi k}{2N}\,\frac{1}{r^4}$$

  Strength: k coincident strings/quarks

Interpolating solution

- Exact interpolating solution (BPS):

$$\frac{1}{z(\theta)} = \frac{A}{\sin\theta}\left(\theta - \sin\theta\cos\theta - \tfrac{\pi k}{N}\right)^{1/3}$$

[Callan-Guijosa-Savvidy (1998)]

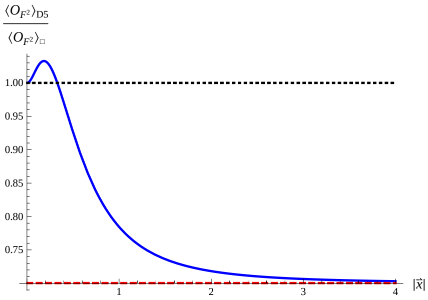
- k fundamental reps. in UV (small z) $\longrightarrow$ $\mathcal{A}_k$ in IR (large z)

- For small z (UV)

$$\theta(z) \simeq \pi - A\,z\,\ldots$$

Interpreted as a VEV for the $\Delta = 1$ operator in CFT_1

Interpolating $\frac{1}{N}\langle \mathrm{Tr} F^2 \rangle$



- Interpretation: k fundamental sources screened to antisymmetric representation. [SPK-Silvani 1611.06033]

Symmetric represention $\mathcal{S}_k$ and deformation

- D3-brane embedding for rep. $\mathcal{S}_k$: $AdS_2 \times S^2$

$$AdS_5: \quad ds^2 = \frac{dz^2}{z^2} + \frac{1}{z^2}\left(-dt^2 + d\rho^2 + \rho^2 d\Omega_2^2\right)$$

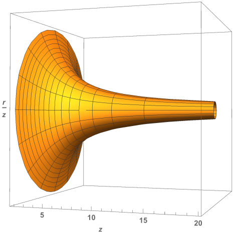
$$D3: \quad (t, z, \rho, \Omega_2) \qquad \boxed{\rho = \kappa\, z} \qquad \kappa \equiv \frac{\sqrt{\lambda}k}{4N}$$

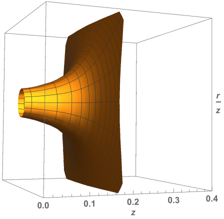
- "Breathing mode" of S^2 has $m^2 = 0$: $\Delta = 1$ in CFT_1

- Exact non-conformal (BPS) embedding: $\boxed{\rho = \frac{\kappa\, z}{1 + a\,\kappa\, z}}$

Two non-conformal solutions

- $\underline{a > 0}$: Rep. $\mathcal{S}_k$, small az (UV) $\longrightarrow$ k fundamental quarks, large az (IR)



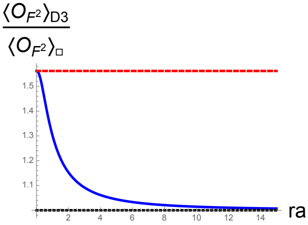


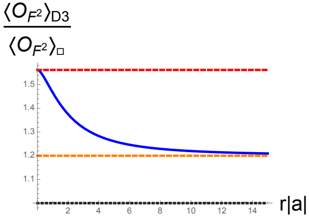
- $\underline{a < 0}$: Rep. $\mathcal{S}_k$ heavy quark in a Coulomb phase with $SU(N) \to U(1) \times SU(N-1)$

Interpolating solution

- Small z: $\frac{\rho}{z} = \kappa - a\kappa^2 z \ldots$

  Flow triggered by VEV of $\Delta = 1$ operator in UV





- UV: $\frac{1}{N}\langle \mathrm{Tr}\, F^2 \rangle \rightarrow \frac{\sqrt{2}}{4\pi}\,\kappa\sqrt{1+\kappa^2}\,\rho^{-4}$ (Fiol, Garolera, Lewkowycz (2012))

  IR: $(a > 0) \quad \rightarrow \frac{\sqrt{2}}{16\pi}\sqrt{\lambda}\frac{k}{N}\,\rho^{-4} \qquad a < 0 : \quad \rightarrow \frac{\sqrt{2}}{4\pi}\,\kappa^2\,\rho^{-4}$

- $(a > 0)$ soln: Symmetric rep. source "dissociating" into fundamental quarks

Brief Summary

- All non-conformal impurities describe flows triggered by VEV for a $\Delta = 1$ mode in CFT_1
- All accompanied by a decrease of the strength of source towards the IR: suggesting a "thinning out" of degrees of freedom
- We turn to an alternative measure: The contribution from the impurity to the Entanglement Entropy of a spherical region enclosing it

Impurity Entanglement Entropy

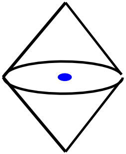
- EE measures the degree of quantum entanglement in a given QFT state
- For a subsystem A, defined via the reduced density matrix ρ_A
- $S_{\mathrm{EE}}(A) = -\mathrm{Tr}\rho_A \ln \rho_A$
- Computed using the replica trick:

$$S_{\mathrm{EE}}(A) = -\lim_{n\to 1} \frac{\partial}{\partial n} \mathrm{Tr}\rho_A^n$$

- The holographic Ryu-Takayanagi minimal area surface prescription cannot be immediately applied for external probes. Need backreaction from the probes.

Generalized gravitational entropy

- Implement replica trick in the bulk. [Lewkowycz-Maldacena (2013)]
- We are interested in the EE excess due to impurity for a spherical region $\mathcal{B}$ of radius r centred around the impurity.



- Map the causal development $\mathcal{D}$ of the $\mathcal{B}$ to the Rindler wedge, which is conformal to hyperbolic space H_3 at temperature $\beta = \frac{1}{2\pi}$
- The bulk dual of $\mathcal{D}$ is hyperbolically sliced AdS_5 with horizon and Hawking temperature β.

Impurity EE

- $$ds^2 = \frac{d\rho^2}{f_n(\rho)} + f_n(\rho)\, d\tau^2 + \rho^2 \left(du^2 + \sinh^2 u\, d\Omega_2^2\right)$$

  $$f_n(\rho) = \rho^2 - 1 - \frac{\rho_+(\rho_+^2 - 1)}{\rho^2}$$

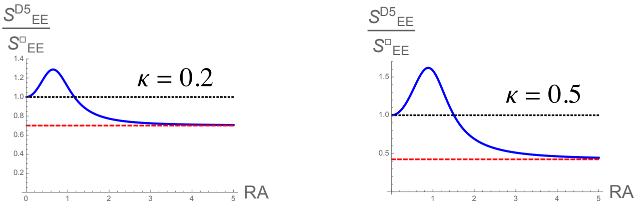
  $$\rho_+ = \frac{1}{4n}\left(1 + \sqrt{1+8n^2}\right).$$

- The excess EE due to impurity given by varying probe action wrt β:

  $$S_{EE}^{\rm imp}(r) = \lim_{\beta\to 2\pi} \beta\partial_\beta S_{\rm D3,D5}(\beta)$$

  (Importantly, one may use the embedding at $\beta = 2\pi$ to evaluate the action above [Karch, Uhlemann (2014)])

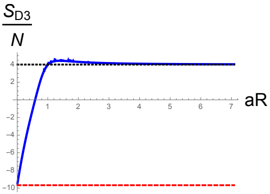
Impurity EE for screening to $\mathcal{A}_k$



$$\left.\frac{S_{\mathrm{D5}}}{k\,S_{\square}}\right|_{Ar\to 0} = 1\,, \qquad \left.\frac{S_{\mathrm{D5}}}{k\,S_{\square}}\right|_{Ar\to\infty} = \frac{2N}{3\pi k}\sin^3\theta_k\,.$$

• Non-monotonic, settling at lower value in deep IR, and qualitatively tracking $\langle \mathrm{Tr} F^2\rangle$

Impurity EE for flow from $\mathcal{S}_k$

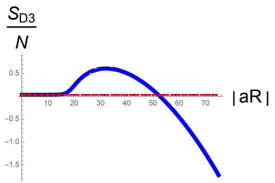


$$S_{\rm D3}\big|_{ar\to 0} = S_{\mathcal{S}_k} = N\left(\sinh^{-1}\kappa - \frac{1}{3}\kappa\sqrt{1+\kappa^2}\right)$$

$$S_{\rm D3}\big|_{ar\to\infty} = kS_{\Box} = \frac{2}{3}N\kappa$$

- Non-monotonic, not positive definite, settling at higher IR value, disagrees qualitatively with $\langle {\rm Tr} F^2\rangle$

Coulomb Branch $\mathcal{S}_k$



$$S_{\rm D3}\Big|_{ar\gg 1} = N\left[-\frac{1}{3}(|a|r\tilde{\kappa})^2 - |a|r + \frac{2}{3}\ln\left(2|a|r\tilde{\kappa}\right)\right]\,. \qquad (1)$$

Impurity Quantum Mechanics

- On general grounds and specifically by considering brane-intersections impurity theory has the form (Gomis-Passerini (2006)):

$$S_{\rm imp} = \quad S_{\mathcal{N}=4} + \int dt \left[i\bar{\chi}_m \partial_t \chi^m + \bar{\chi}_m \left(A_0 + \hat{n}^I \, \phi^I \right)^m{}_n \chi^n + \right.$$

$$\left. + \mu \left(\bar{\chi}_m \chi^m - k \right) \right] .$$

  $\{\chi_m\}$ are N fermions (bosons) for antisymmetric (symmetric) Wilson lines

- The bulk field $(A_0 + \hat{n}^I \, \phi^I)$ has constant propagator (Erickson-Semenoff-Zarembo; Drukker-Gross) for circular Wilson line (and for Polyakov loops)

$$D(t - t') \sim \frac{\lambda}{N} \qquad (2)$$

Impurity QM

- Integrate out $(A_0 + \hat{n}^I \phi^I)$ exactly:

$$S_{\rm imp} = \int_0^\beta d\tau \, (\bar{\chi}_m \, \partial_\tau \chi^m + \tag{3}$$

$$\frac{1}{2} \int_0^\beta d\tau' \, D_\beta(\tau - \tau') \, \bar{\chi}_m(\tau) \, \chi^n(\tau) \, \bar{\chi}_n(\tau') \, \chi^m(\tau') \Big) \, .$$

- This theory can be solved exactly at large-N, k and at large λ to yield the action $\sim \sin^3 \theta_k$ for circular loops. Sachdev (2010); Mueck (2011)

Further Questions

- Use the impurity QM as a starting point to understand deformations and "flows". In particular, the interpolation between the symmetric representation and k fundamentals.
- All flows discussed are puzzling: originate from a "VEV" in the UV of the impurity theory. Spontaneous breaking of (conformal) symmetry in QM should not be possible.
- The impurity contribution to EE indicates the absence of a "g-theorem" for higher codimension impurities. It would be interesting to understand or prove this using purely holographic methods.