Isotropization in the Color Glass Condensate

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Holoquark, Santiago de Compostela, July 2018

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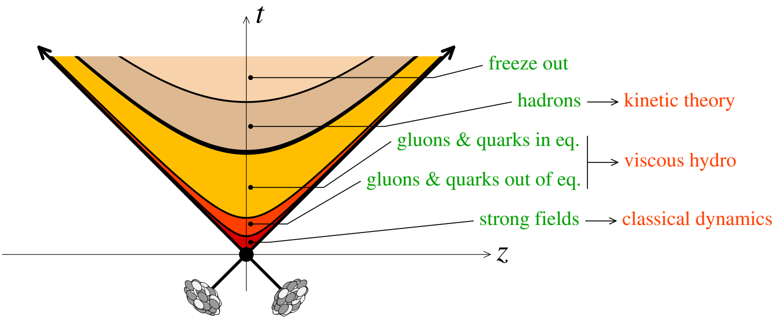
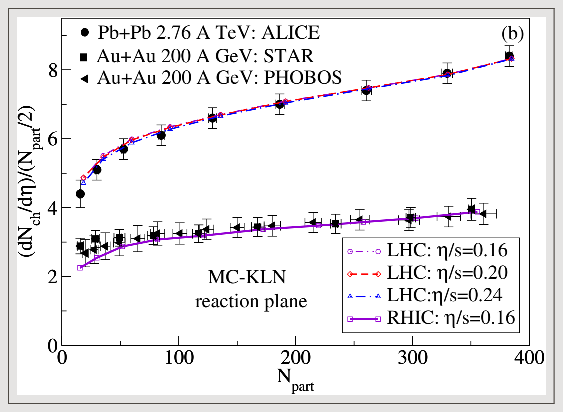
STAGES OF A NUCLEUS-NUCLEUS COLLISION



- Hydrodynamics successful at describing the bulk evolution
- In this talk : **Pre-hydrodynamical evolution**

Evidence for hydrodynamical expansion



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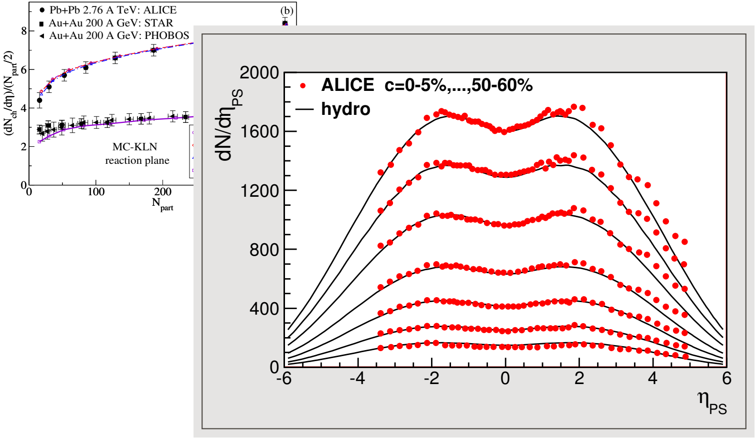

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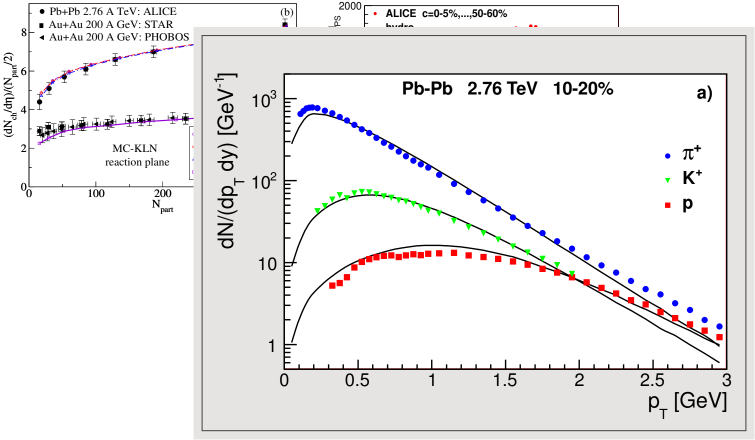

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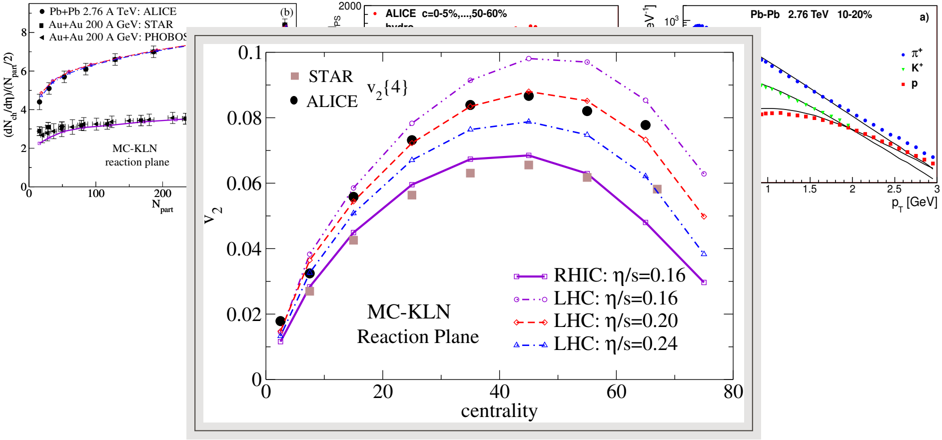

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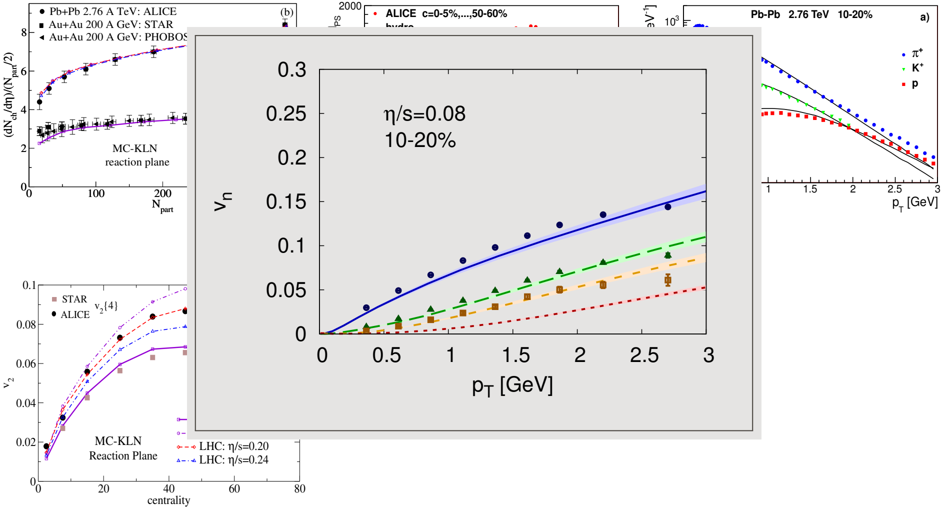
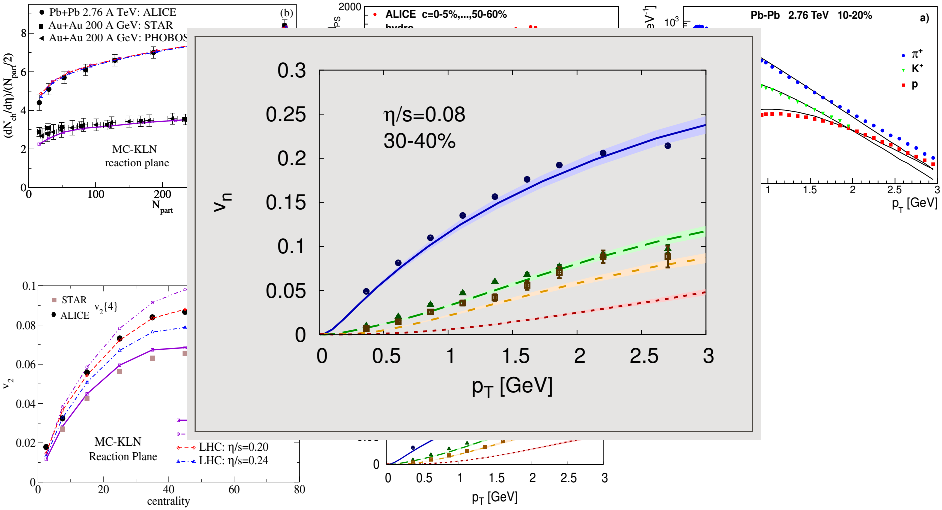
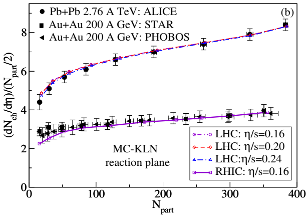
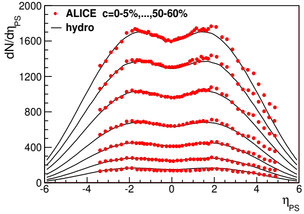


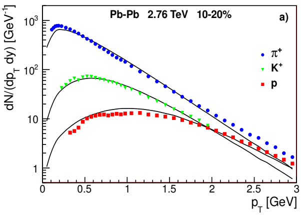

η/s=0.08
30-40%
v_n
p_T [GeV]
MC-KLN
reaction plane
MC-KLN
Reaction Plane
LHC: η/s=0.20
LHC: η/s=0.24
ALICE c=0-5%,...,50-60%
Pb-Pb 2.76 TeV 10-20%



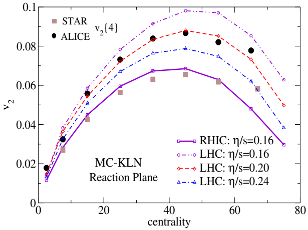
Pb+Pb 2.76 A TeV: ALICE
Au+Au 200 A GeV: STAR
Au+Au 200 A GeV: PHOBOS
(b)
(dN_ch/dη)/(N_part/2)
MC-KLN
reaction plane
LHC: η/s=0.16
LHC: η/s=0.20
LHC:η/s=0.24
RHIC: η/s=0.16
N_part



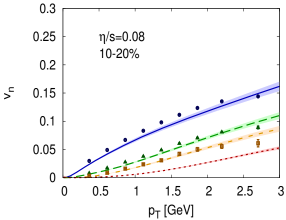
ALICE c=0-5%,...,50-60%
hydro
dN/dη_PS
η_PS



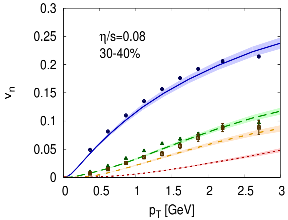
Pb-Pb 2.76 TeV 10-20%
a)
π+
K+
p
dN/(dp_T dy) [GeV^-1]
p_T [GeV]



STAR
ALICE
v2{4}
v2
MC-KLN
Reaction Plane
RHIC: η/s=0.16
LHC: η/s=0.16
LHC: η/s=0.20
LHC: η/s=0.24
centrality



η/s=0.08
10-20%
v_n
p_T [GeV]



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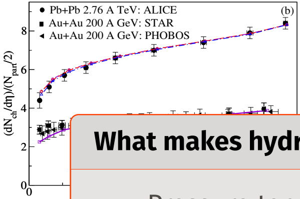
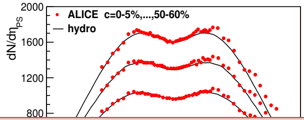
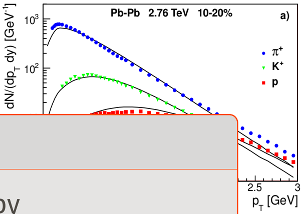


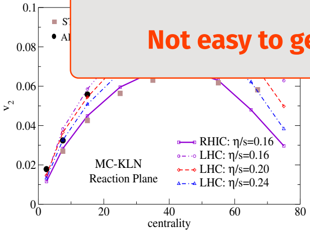


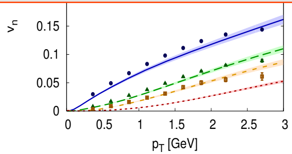


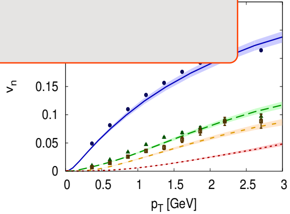
What makes hydrodynamics work so well?

- Pressure tensor “not too far” from isotropy
- Low viscosity (compared to entropy density)

Not easy to get in QCD...

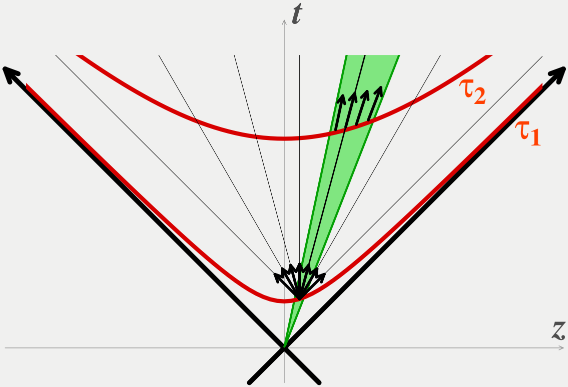






Competition between expansion and interactions

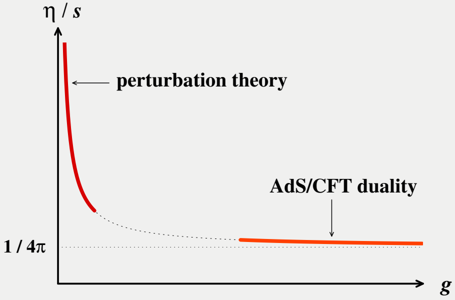
- Very different from isotropization in a box
- Sustained interactions are needed for isotropy to persist despite the expansion



SHEAR VISCOSITY AT WEAK AND STRONG COUPLING (IN EQUILIBRIUM)

Weak coupling result [Arnold, Moore, Yaffe (2000)]

$$\frac{\eta}{s} \approx \frac{5.1}{g^4 \ln\left(\frac{2.4}{g}\right)}$$



Weak coupling result [Arnold, Moore, Yaffe (2000)]

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Note :

- Weak coupling result is for equilibrium distributions $f_k \sim 1$
- Ruled out: $(g \ll 1 \text{ AND } f_k \sim 1)$
- Not ruled out: $\underbrace{(g \gg 1 \text{ AND } f_k \sim 1)}_{\text{Holography}}$ OR $\underbrace{(g \ll 1 \text{ AND } f_k \gg 1)}_{\text{CGC}}$

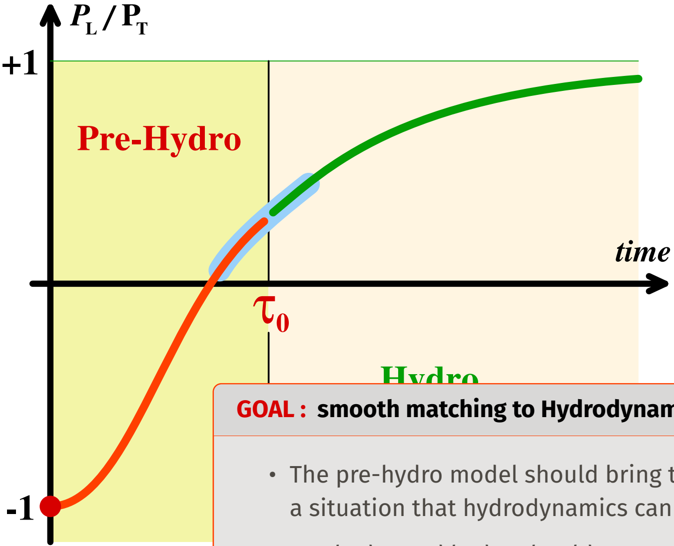
Kinetic theory wisdom :

$$\frac{\eta}{s} \sim \frac{\text{mean free path}}{\text{de Broglie wavelength}}$$

- (**de Broglie wavelength**)$^{-1} \sim Q$
- (**mean free path**)$^{-1} \sim \underbrace{g^4 Q^{-2}}_{\text{cross section}} \times \underbrace{\int_k f_k}_{\text{density}} \underbrace{(1 + f_k)}_{\text{Bose enhancement}}$

If $g \ll 1$ **but** $f_k \sim g^{-2}$ **(weakly coupled, but strongly interacting)**

$$\frac{\eta}{s} \sim g^0$$

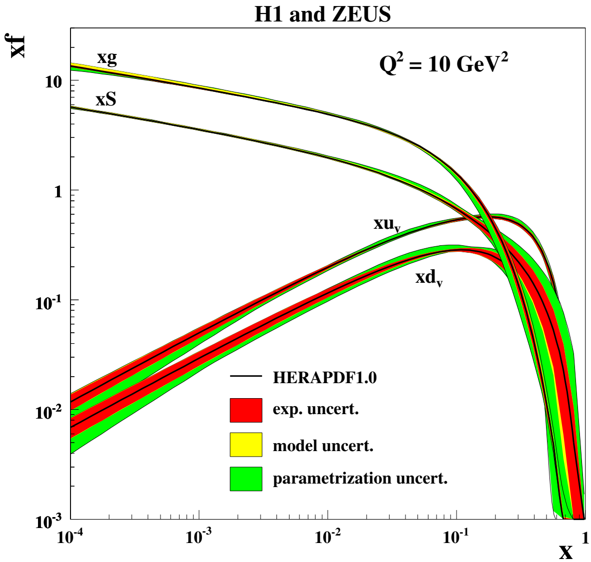


GOAL : smooth matching to Hydrodynamics

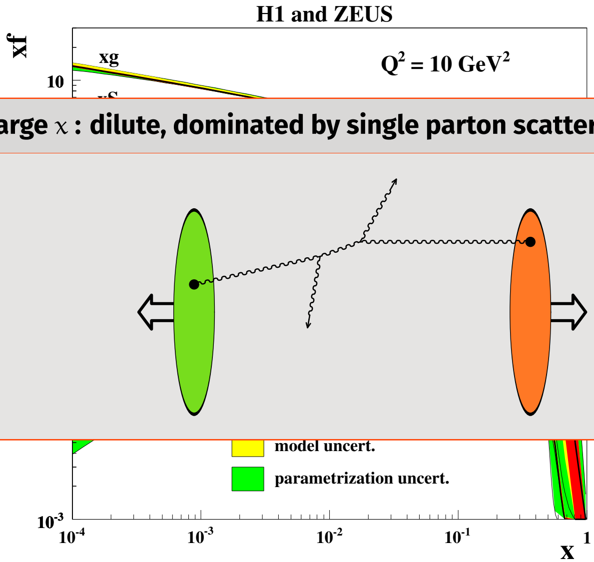
- The pre-hydro model should bring the system to a situation that hydrodynamics can handle
- Pre-hydro and hydro should agree over some range of time $\Rightarrow$ no τ_0 dependence
- Description as close as possible to QCD

QCD description up to $\tau = 0^+$

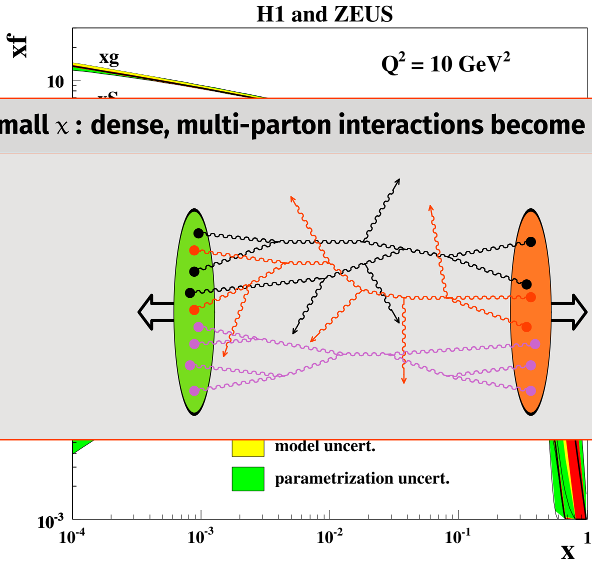
Parton distributions in a nucleon



Parton distributions in a nucleon



Large x : dilute, dominated by single parton scattering



Small x : dense, multi-parton interactions become likely

- When their occupation number becomes large, gluons can recombine :

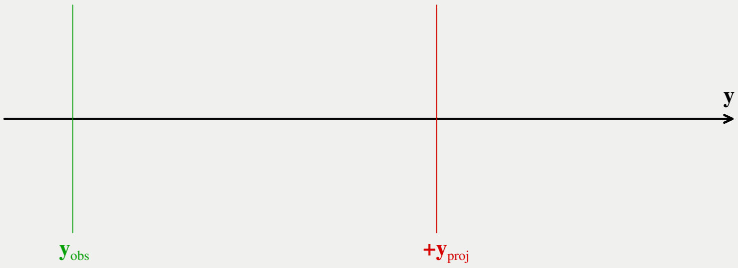
Gluon Saturation

Saturation criterion **[Gribov, Levin, Ryskin (1983)]**

$$\underbrace{\alpha_s Q^{-2}}_{\sigma_{gg\to g}} \times \underbrace{A^{-2/3} xG(x,Q^2)}_{\text{surface density}} \geq 1$$

$$Q^2 \leq \underbrace{Q_s^2 \equiv \frac{\alpha_s xG(x,Q_s^2)}{A^{2/3}}}_{\text{(saturation momentum)}^2} \sim A^{1/3} x^{-\lambda} \quad (\lambda \approx 0.25)$$

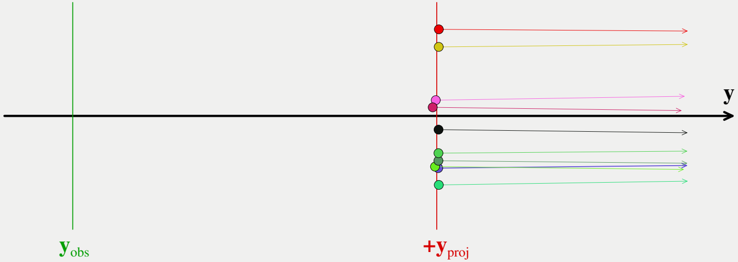
DEGREES OF FREEDOM AT VARIOUS RAPIDITIES $(y \sim \ln(p_z))$



McLerran-Venugopalan model :

- Fast partons : frozen dynamics, negligible $p_\perp$ $\Rightarrow$ classical current
- Slow partons : evolve with time $\Rightarrow$ gauge fields

Degrees of freedom at various rapidities ($y \sim \ln(p_z)$)



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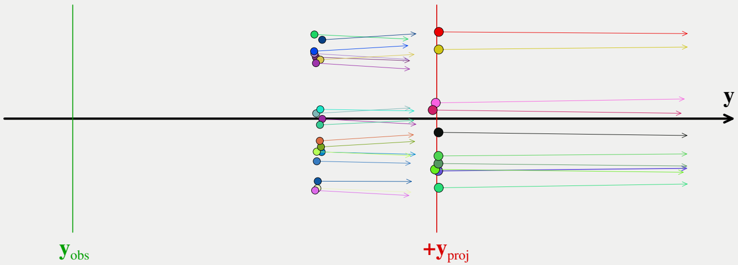

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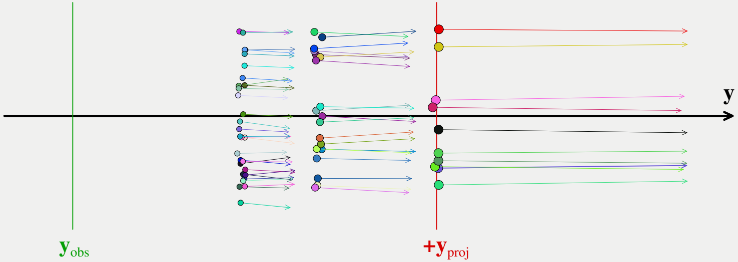

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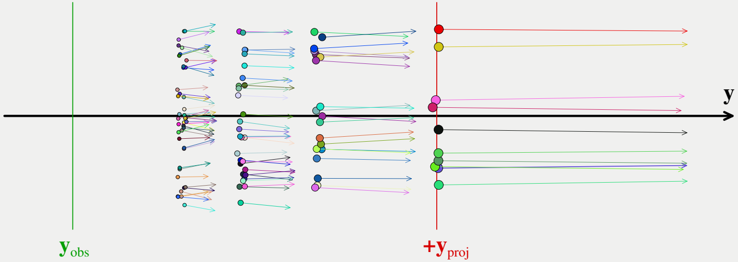

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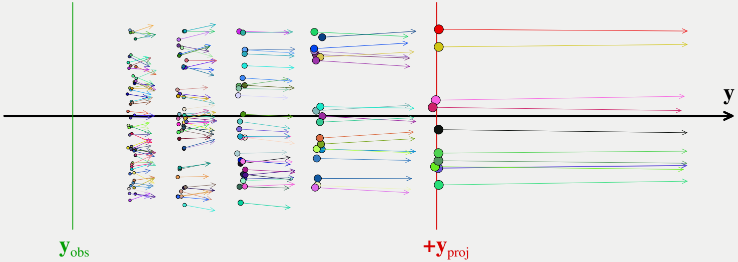

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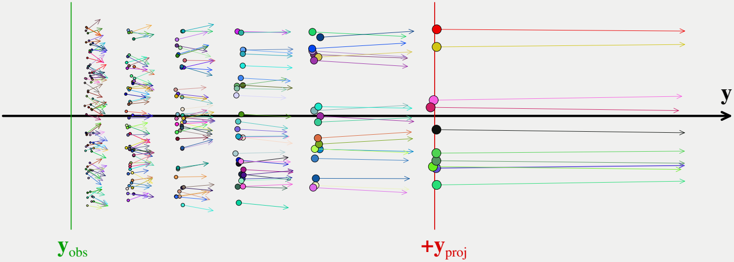

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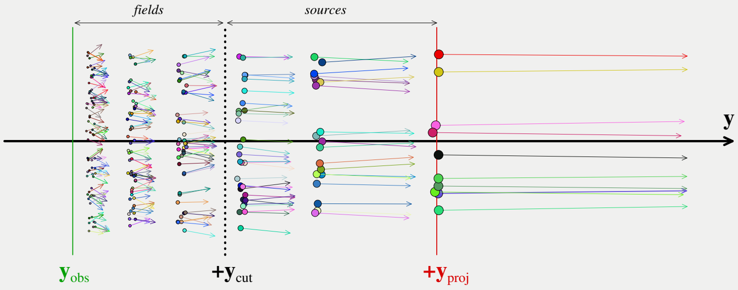

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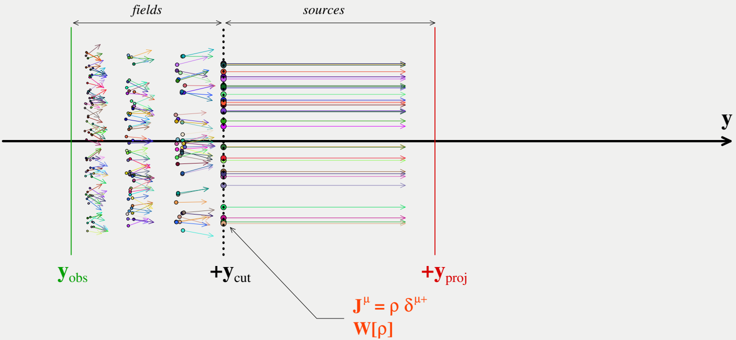

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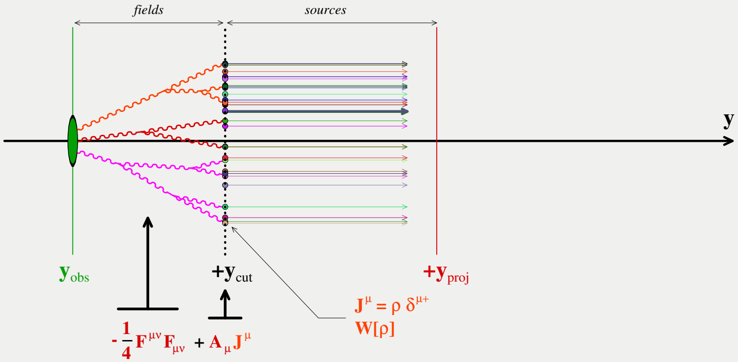

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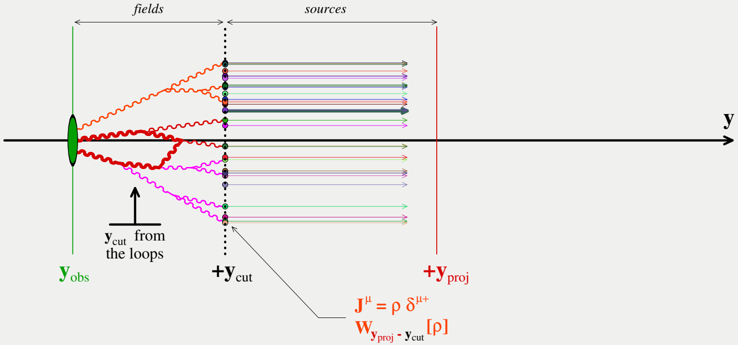
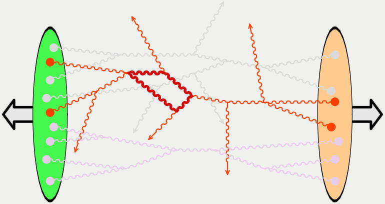
Cancellation of the cutoff dependence



- The probability density $W[\rho]$ changes with the cutoff
- Loop corrections cancel the cutoff dependence from $W[\rho]$

Power counting in the saturated regime

$$\mathcal{S} = \underbrace{-\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}}_{\text{slow}} + \underbrace{\int (J_1^\mu + J_2^\mu)}_{\text{fast}} A_\mu$$



In the saturated regime: $J^\mu \sim g^{-1}$, $A^\mu \sim g^{-1}$, $f_k \sim g^{-2}$

$$T^{\mu\nu} \sim \frac{1}{g^2}\left[c_0 + c_1\, g^2 + c_2\, g^4 + \cdots\right]$$

$c_0 \equiv$ tree level, $c_1 \equiv$ one loop, etc...

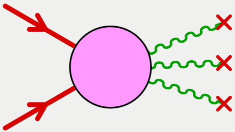
- Average particle multiplicity $\sim 1/g^2 \gg 1$
- Probability of a given final state $\sim \exp(-\frac{\#}{g^2}) \ll 1$

  $\Longrightarrow$ not very useful

- Inclusive observables :
  average of some quantity over all possible final states

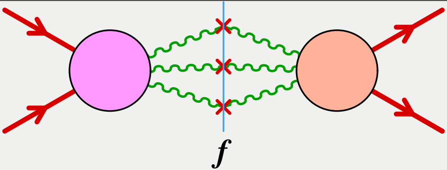
$$\langle \mathcal{O} \rangle \equiv \sum_{\substack{\text{all final} \\ \text{states } f}} \mathcal{P}(AA \to \mathbf{f})\; \mathcal{O}(\mathbf{f})$$

Schwinger-Keldysh formalism : technique to perform the sum over final states without computing the individual transition probabilities $\mathcal{P}(AA \to \mathbf{f})$



Time-ordered perturbation theory :

$$G_{++}(p) = \frac{i}{p^2 + i\epsilon}$$

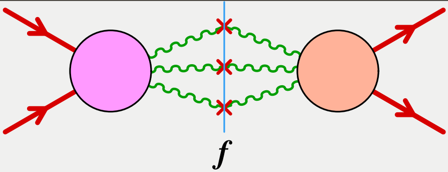


Time-ordered perturbation theory :

$$G_{++}(p) = \frac{i}{p^2 + i\epsilon}$$

Anti time-ordered perturbation theory :

$$G_{--}(p) = \frac{-i}{p^2 - i\epsilon}$$



Time-ordered perturbation theory :

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Anti time-ordered perturbation theory :

$$G_{--}(p) = \frac{-i}{p^2 - i\epsilon}$$

Schwinger-Keldysh formalism :

- Across the cut : $G_{+-}(p) \equiv 2\pi\,\theta(-p^0)\,\delta(p^2)$
- Final state sum : sum over all the assignments of the labels $+$ and $-$ to vertices and sources

Leading Order

- Leading Order = sum of all tree diagrams

  Expressible in terms of classical solutions of Yang-Mills equations :

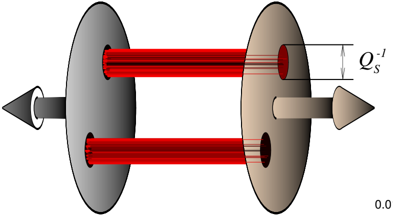
$$\left[\mathcal{D}_\mu, \mathcal{F}^{\mu\nu}\right] = J_1^\nu + J_2^\nu$$

- Initial condition : $\lim_{x^0 \to -\infty} \mathcal{A}^\mu(x) = 0$

Components of the energy-momentum tensor

$$T^{00}_{\text{LO}} = \frac{1}{2}\big[\underbrace{\mathbf{E}^2 + \mathbf{B}^2}_{\text{class. fields}}\big] \qquad T^{0i}_{\text{LO}} = \left[\mathbf{E}\times\mathbf{B}\right]^i$$

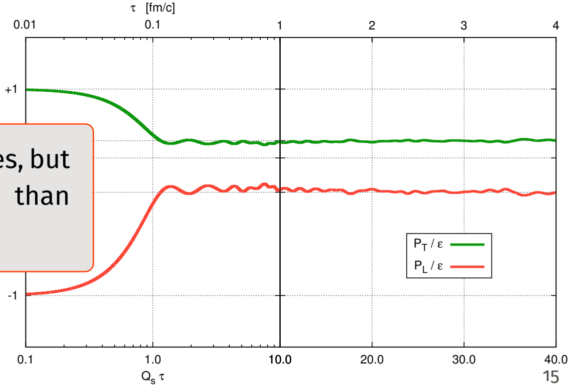
$$T^{ij}_{\text{LO}} = \frac{\delta^{ij}}{2}\left[\mathbf{E}^2 + \mathbf{B}^2\right] - \left[\mathbf{E}^i\mathbf{E}^j + \mathbf{B}^i\mathbf{B}^j\right]$$



At $\tau = 0^+$

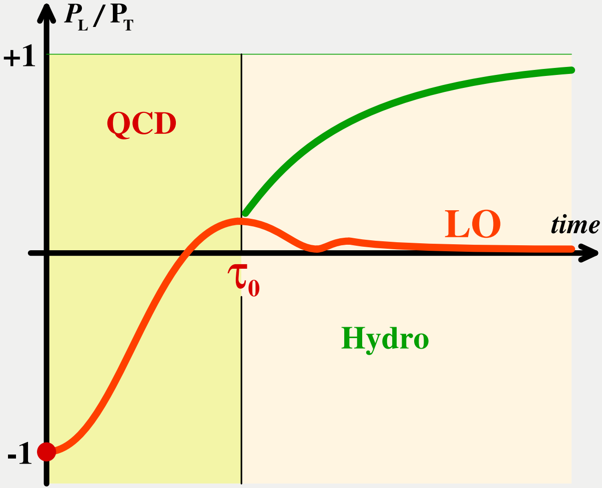
$$\mathbf{E} \parallel \mathbf{B} \parallel \hat{z}$$

$$P_T = \epsilon, \; P_L = -\epsilon$$



P_L rises to positive values, but remains much smaller than P_T (free streaming)

LO : UNSATISFACTORY MATCHING TO HYDRODYNAMICS



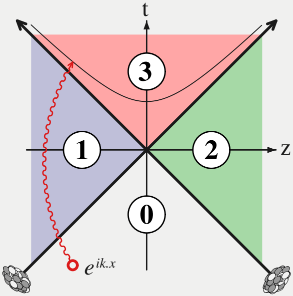
Next-to-Leading Order

- LO : classical field $\mathcal{A}^\mu_{\rm LO} \sim Q_s/g$
- NLO : one-loop in a non-trivial background
  - Gaussian fluctuations $\alpha^\mu \sim Q_s$ on top of $\mathcal{A}^\mu_{\rm LO}$
  - Variance $\boldsymbol{\sigma}(\boldsymbol{u},\boldsymbol{v})$ of the fluctuations known analytically at $\tau = 0^+$

$$\boldsymbol{\sigma}(\boldsymbol{u},\boldsymbol{v}) = \frac{1}{2}\int_{\text{modes }\mathbf{k}} \boldsymbol{\alpha}_k(\boldsymbol{u})\boldsymbol{\alpha}^*_k(\boldsymbol{v})$$

$$\left[\mathcal{D}_\rho\mathcal{D}^\rho\delta^\nu_\mu - \mathcal{D}_\mu\mathcal{D}^\nu + ig\,\mathcal{F}_\mu{}^\nu\right]\boldsymbol{\alpha}^\mu_k = 0$$

$$\lim_{x^0\to-\infty} \boldsymbol{\alpha}^\mu_k(x) = \epsilon^\mu\, e^{ik\cdot x}$$



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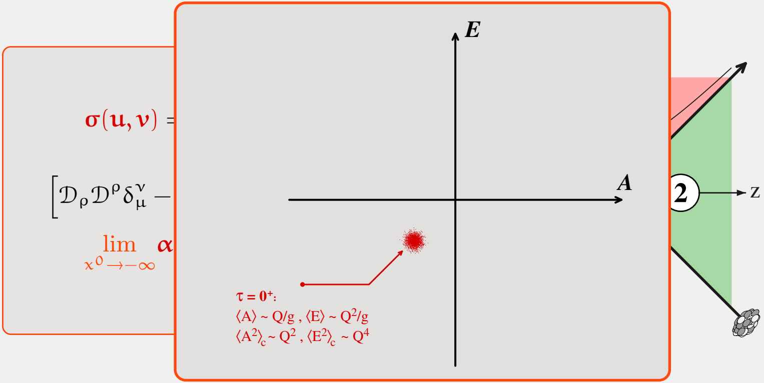
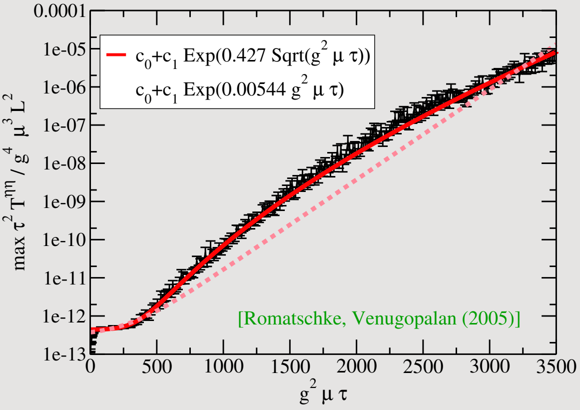
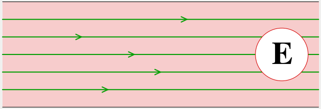


Instabilities and Resummation

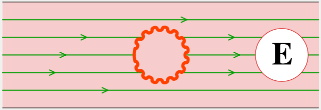
Instability of classical solutions



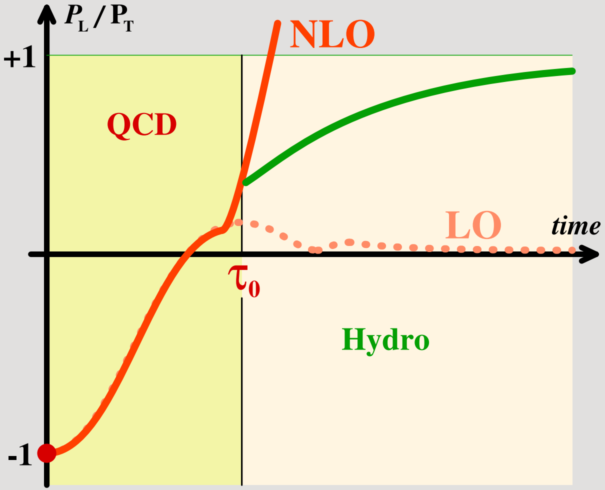
- LO = longitudinal chromo-E and chromo-B fields



- NLO = gluon loop embedded in this field



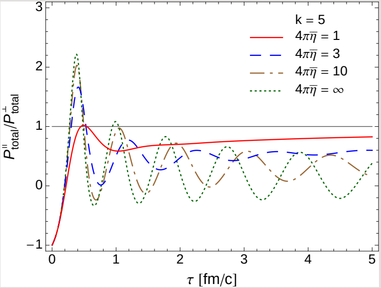
  - instability $\sim$ imaginary part of the loop $\sim$ gluon pair production
  - BUT : at NLO, no feedback of the produced gluons on the LO field!


PL / PT
+1
NLO
QCD
LO
time
τ0
Hydro
-1

Color flux tube model : [Ryblewski, Florkowski (2013)]

$$\left(p^\mu \partial_\mu + \underbrace{g F^{\mu\nu} p_\nu}_{\text{Lorentz force}} \partial^\mu_p\right) G = \underbrace{\frac{dN}{d\Gamma}}_{\text{Schwinger}} + \underbrace{C_p[G]}_{\text{collisions}}$$

$$\partial_\mu F^{\mu\nu} = j^\nu \qquad \text{(feedback)}$$



- Field converted into particles by instability
- Nearly constant P_L/P_T
- Ratio depends on $\bar{\eta} \equiv \eta/s$

Inclusive observables at NLO

[FG, Lappi, Venugopalan (2007–2008)]

- Observables at NLO can be obtained from the LO by "fiddling" with the initial condition of the classical field :

$$\mathcal{O}_{\rm NLO} = \frac{\hbar}{2} \int_{\boldsymbol{u},\boldsymbol{v}} \boldsymbol{\sigma}(\boldsymbol{u},\boldsymbol{v}) \, \frac{\delta}{\delta \mathcal{A}_{\rm ini}(\boldsymbol{u})} \frac{\delta}{\delta \mathcal{A}_{\rm ini}(\boldsymbol{v})} \quad \mathcal{O}_{\rm LO}$$

  - NLO : the time evolution remains classical;

    $\hbar$ only enters in the initial condition

    (NNLO : $\hbar$ starts appearing in the time evolution itself)

Analogue in Quantum Mechanics

- Consider the Liouville–von Neumann equation :

$$i\,\hbar\,\frac{\partial\widehat{\rho}_\tau}{\partial\tau} = \big[\widehat{H},\widehat{\rho}_\tau\big]$$

- Introduce the Wigner transforms :

$$\begin{aligned} W_\tau(\boldsymbol{x},\boldsymbol{p}) &\equiv \int ds\; e^{i\boldsymbol{p}\cdot\boldsymbol{s}}\,\big\langle\boldsymbol{x}+\frac{\boldsymbol{s}}{2}\big|\widehat{\rho}_\tau\big|\boldsymbol{x}-\frac{\boldsymbol{s}}{2}\big\rangle \\ \mathcal{H}(\boldsymbol{x},\boldsymbol{p}) &\equiv \int ds\; e^{i\boldsymbol{p}\cdot\boldsymbol{s}}\,\big\langle\boldsymbol{x}+\frac{\boldsymbol{s}}{2}\big|\widehat{H}\big|\boldsymbol{x}-\frac{\boldsymbol{s}}{2}\big\rangle \end{aligned}$$

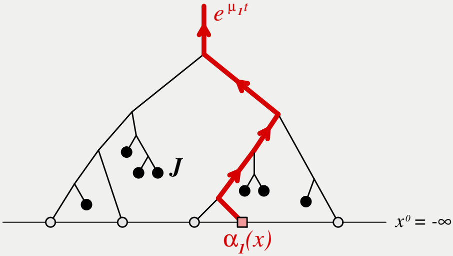
- LvN equation is equivalent to Moyal-Groenewold equation

$$\begin{aligned} \frac{\partial W_\tau}{\partial\tau} &= \mathcal{H}(\boldsymbol{x},\boldsymbol{p})\,\frac{2}{i\,\hbar}\,\sin\left(\frac{i\,\hbar}{2}\big(\overleftarrow{\partial}_{\boldsymbol{p}}\overrightarrow{\partial}_{\boldsymbol{x}}-\overleftarrow{\partial}_{\boldsymbol{x}}\overrightarrow{\partial}_{\boldsymbol{p}}\big)\right)W_\tau(\boldsymbol{x},\boldsymbol{p}) \\ &= \underbrace{\{\mathcal{H},W_\tau\}}_{\text{Poisson bracket}} + \underbrace{\mathcal{O}(\hbar^2)}_{\text{deviation from classical dynamics}} \end{aligned}$$

IMPROVED POWER COUNTING

- For an unstable mode:

$$\alpha_k(x) \underset{x^0\to+\infty}{\sim} e^{\mu_k x^0} \qquad (\mu_k = \text{Lyapunov exponent})$$



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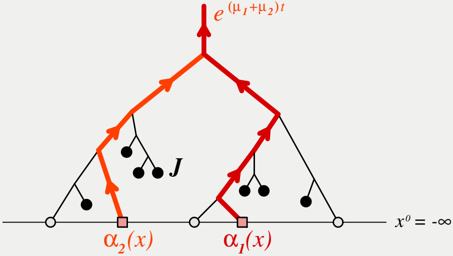

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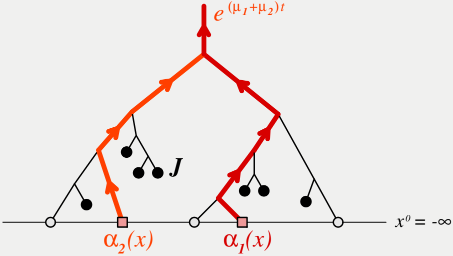
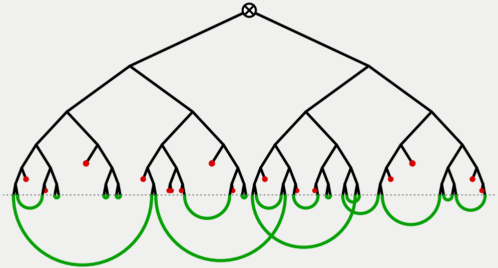


- 1 loop : $g^2 \, \hbar \, e^{2\mu_k t}$
- n loops : $\left(g^2 \, \hbar \, e^{2\mu_k t}\right)^n$

Resummation of the leading terms



$$\mathcal{O}_{\text{resummed}} \equiv \exp\left[\frac{\hbar}{2}\int_{\mathbf{u},\mathbf{v}} \boldsymbol{\sigma}(\mathbf{u},\mathbf{v})\,\frac{\delta}{\delta\mathcal{A}_{\text{ini}}(\mathbf{u})}\frac{\delta}{\delta\mathcal{A}_{\text{ini}}(\mathbf{v})}\right]\;\mathcal{O}_{\text{LO}}$$

$\mathcal{O}_{\text{resummed}} = \mathcal{O}_{\text{LO}} + \mathcal{O}_{\text{NLO}} +$ subset of all higher orders

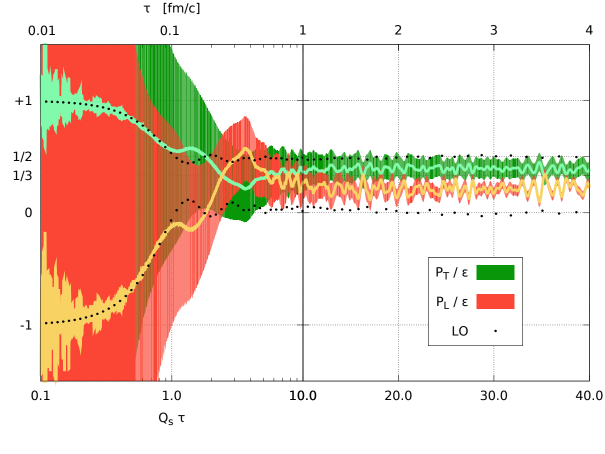
$$\exp\left[\frac{\hbar}{2}\int_{u,v} \underbrace{\sigma(u,v)\,\frac{\delta}{\delta\mathcal{A}_{\rm ini}(u)}\,\frac{\delta}{\delta\mathcal{A}_{\rm ini}(v)}}_{\sim\ \text{Laplacian}}\right]\ \mathcal{O}_{\rm LO}\left[\mathcal{A}_{\rm ini}\right]$$

$$=\int\left[D\alpha(u)\right]\ \exp\left[-\frac{1}{2\,\hbar}\int_{u,v}\alpha(u)\,\sigma^{-1}(u,v)\,\alpha(v)\right]\ \mathcal{O}_{\rm LO}\left[\mathcal{A}_{\rm ini}+\alpha\right]$$

- In this resummation, the observable is obtained as an average over classical fields with fluctuating initial conditions
- The variance of the fluctuations ($\hbar\,\sigma$) is prescribed by the NLO

Evolution at small coupling : $g = 0.5 \quad (\lambda \equiv g^2 N_c = 0.5)$

[Epelbaum, FG (2013)]



Is it possible to start from a decohered state ?

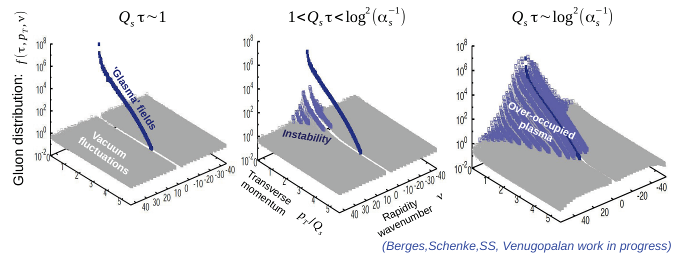
QCD at $\tau = 0^+$: coherent initial state

$$\mathcal{A} = \mathcal{A}_{\rm LO} + \int_{\mathbf{p}} c_{\mathbf{p}}\ \alpha_{\mathbf{p}} \qquad \big\langle c_{\mathbf{p}} c_{\mathbf{p}'} \big\rangle \sim \frac{1}{2}\ \delta_{\mathbf{p}\mathbf{p}'}$$

Occupation number :

$$\big\langle \widetilde{\mathcal{A}}\widetilde{\mathcal{A}}^* \big\rangle_{\tau=0^+} = \underbrace{\widetilde{\mathcal{A}}_{\rm LO}\widetilde{\mathcal{A}}^*_{\rm LO}}_{\sim\delta(p_z)f(p_\perp)} + \frac{1}{2}$$

Is it possible to start from a decohered state ?



- At $\tau \gtrsim Q_s^{-1}$, large occupation in a broad range of $p_z, p_\perp$

Incoherent distribution of particles :

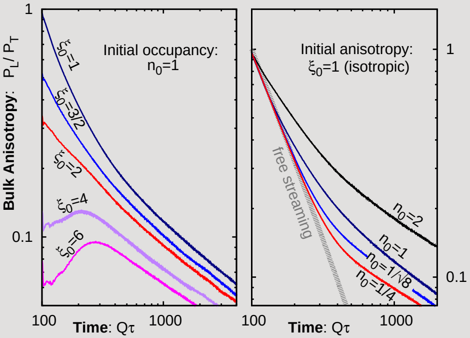
$$A = \int_{\mathbf{p}} c_{\mathbf{p}}\ \alpha_{\mathbf{p}} \qquad \left\langle c_{\mathbf{p}} c_{\mathbf{p}'} \right\rangle \sim \delta_{\mathbf{p}\mathbf{p}'} \left[\frac{1}{2} + f_0(\mathbf{p})\right]$$

$\frac{1}{2} \iff$ zero point fluctuations

$f_0(\mathbf{p}) \iff$ initial particle distribution ($\sim g^{-2}$)

If $f_0(\mathbf{p}) \gg 1$, approximate $\frac{1}{2} + f_0 \ \rightarrow\ f_0$?

[Berges, Boguslavski, Schlichting, Venugopalan (2013)]



- No dependence on the coupling (can be scaled out)
- $P_T/P_L \sim \tau^{-2/3}$ (approx breaks at $Q\tau \sim \alpha_s^{-3/2}$)

Classical Statistical Approximation

$$\langle \mathcal{O} \rangle = \int \left[D\phi_+ D\phi_- \right] \, \mathcal{O}[\phi] \; e^{i(S[\phi_+]-S[\phi_-])}$$

- ϕ_+ = amplitude ϕ_- = conjugate amplitude
- $\phi_+ - \phi_-$ = quantum interference

- Introduce : $\phi_1 \equiv \phi_+ - \phi_-$, $\phi_2 \equiv \frac{1}{2}(\phi_+ + \phi_-)$

$$\underbrace{S[\phi_+] - S[\phi_-]}_{\text{odd in } \phi_1} = \phi_1 \cdot \frac{\delta S[\phi_2]}{\delta \phi_2} + \text{terms cubic in } \phi_1$$

- Strong field regime : $\phi_\pm$ large, but $\phi_+ - \phi_-$ small

  $\rightarrow$ Neglect the terms cubic in ϕ_1

  $D\phi_1 \;\rightarrow\;$ classical Euler-Lagrange equation for ϕ_2

- Remaining fluctuations in the initial condition for ϕ_2

Schwinger-Keldysh perturbation theory

$$G_{++}(p) = \frac{i}{p^2 + i\epsilon} + 2\pi\, f_0(p)\delta(p^2) \qquad G_{--}(p) = \Big[G^*_{++}(p)\Big]^*$$

$$G_{-+}(p) = 2\pi\,(\theta(p^0) + f_0(p))\delta(p^2) \qquad G_{+-}(p) = G_{-+}(-p)$$

$$\Gamma_{++++} = -ig^2 \qquad \Gamma_{----} = +ig^2$$

After rotation $\phi_\pm \;\to\; \phi_{1,2}$ **:**

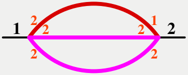
$$G_{21}(p) = \frac{i}{p^2 + ip^0\epsilon} \qquad G_{12}(p) = \frac{i}{p^2 - ip^0\epsilon}$$

$$G_{22}(p) = 2\pi\,(\tfrac{1}{2} + f_0(p))\delta(p^2) \qquad G_{11}(p) = 0$$

$$\Gamma_{1222} = -ig^2 \qquad \Gamma_{1112} = -\tfrac{i}{4}g^2$$

- **Weak CSA :** drop Γ_{1112}
- **Strong CSA :** drop Γ_{1112} AND the $1/2$ in $\frac{1}{2} + f_0(p)$

- CSA $\neq$ underlying theory at 2-loops and beyond
- Vacuum fluctuations make the **Weak CSA** non-renormalizable
  Example of problematic graph :

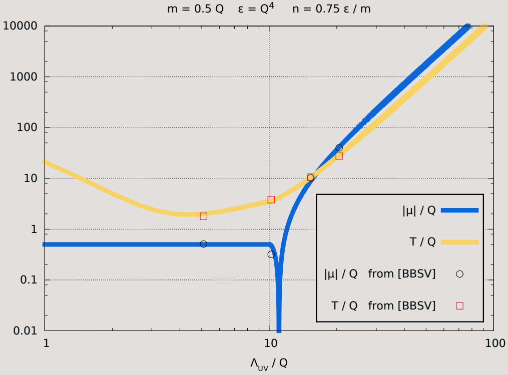
Im 

$$= -\frac{g^4}{1024\pi^3}\left(\Lambda_{\mathrm{UV}}^2 - \frac{2}{3}p^2\right)$$

$\Longrightarrow$ divergence in an operator not present in the Lagrangian

- Strong CSA has no such problem of UV sensitivity

Ultraviolet sensitivity



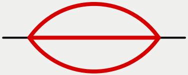
- Weak cutoff dependence if $\Lambda_{\mathrm{UV}} \sim (3-6) \times (\text{physical scales})$

Classical approximations in Kinetic Theory

Dyson-Schwinger equations $\longrightarrow$ Kadanoff-Baym equations $\longrightarrow$ Boltzmann : $p^\mu \partial_\mu f = C_{\mathbf{p}}[f]$

- Collision term in the $\phi_{1,2}$ basis:

$$C_{\mathbf{p}}[f] = \frac{i}{2}\Big[\Sigma_{11}(p) + \big(\tfrac{1}{2} + f(\mathbf{p})\big)\Big(\Sigma_{21}(p) - \Sigma_{12}(p)\Big)\Big]$$



$$\Longrightarrow \quad C_{\mathbf{p}}[f] = \frac{g^4}{4E_{\mathbf{p}}} \int_{\mathbf{k}} \int_{\mathbf{p}'} \int_{\mathbf{k}'} (2\pi)^4 \delta(P + K - P' - K')$$
$$\times \Big[f(\mathbf{p}')f(\mathbf{k}')\big(1 + f(\mathbf{p})\big)\big(1 + f(\mathbf{k})\big)$$
$$- f(\mathbf{p})f(\mathbf{k})\big(1 + f(\mathbf{p}')\big)\big(1 + f(\mathbf{k}')\big)\Big]$$

Weak CSA collision term :

$$C_{\mathbf{p}}[f] \;=\; \frac{g^4}{4E_{\mathbf{p}}}\int_{\mathbf{k}}\int_{\mathbf{p}'}\int_{\mathbf{k}'}(2\pi)^4\delta(P+K-P'-K')$$
$$\times\Big[\big(\tfrac{1}{2}+f(\mathbf{p}')\big)\big(\tfrac{1}{2}+f(\mathbf{k}')\big)\big(\tfrac{1}{2}+f(\mathbf{p})+\tfrac{1}{2}+f(\mathbf{k})\big)$$
$$-\big(\tfrac{1}{2}+f(\mathbf{p})\big)\big(\tfrac{1}{2}+f(\mathbf{k})\big)\big(\tfrac{1}{2}+f(\mathbf{p}')+\tfrac{1}{2}+f(\mathbf{k}')\big)\Big]$$

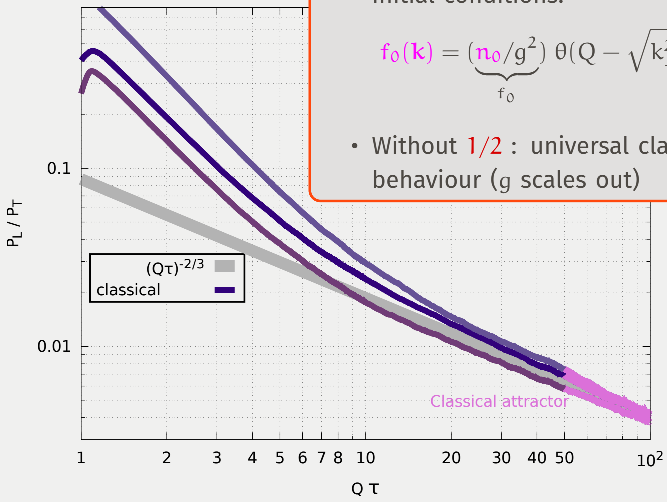
(Terms in f^3 and f^2 correct, but spurious f^1 terms)

Strong CSA : drop also all the $\frac{1}{2}$ (Terms in f^3 correct)

$K')$

$$\times\Big[f(\mathbf{p}')f(\mathbf{k}')\big(1+f(\mathbf{p})\big)\big(1+f(\mathbf{k})\big)$$
$$-f(\mathbf{p})f(\mathbf{k})\big(1+f(\mathbf{p}')\big)\big(1+f(\mathbf{k}')\big)\Big]$$

[Epelbaum, FG, Jeon, Moore, Wu (2015)]

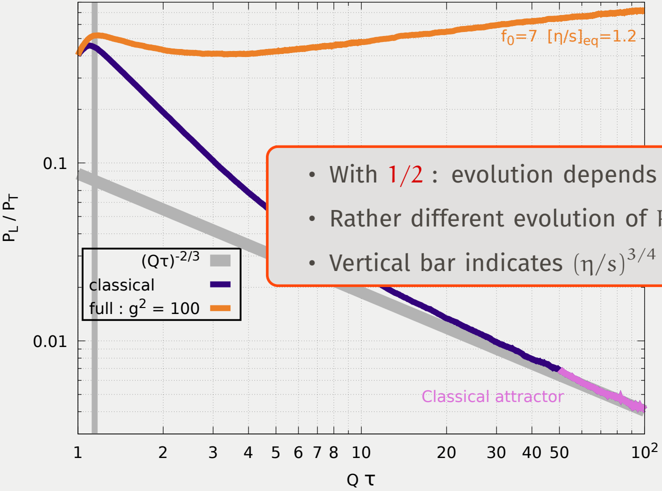


- Initial conditions:

$$f_0(\mathbf{k}) = \underbrace{(n_0/g^2)}_{f_0}\,\theta(Q - \sqrt{k_\perp^2 + \xi_0\,k_z^2})$$

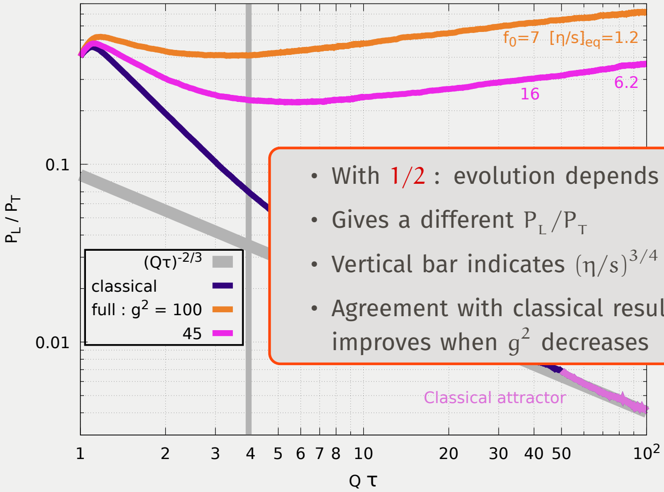
- Without $1/2$: universal classical behaviour (g scales out)

[Epelbaum, FG, Jeon, Moore, Wu (2015)]



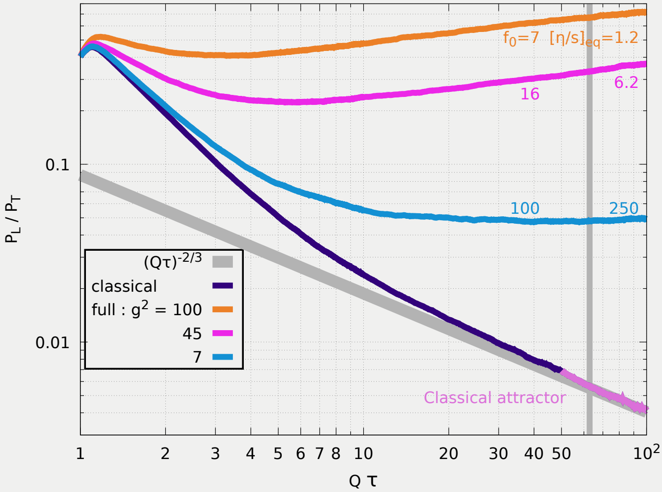
- With 1/2 : evolution depends on g^2
- Rather different evolution of P_L/P_T
- Vertical bar indicates $(\eta/s)^{3/4} \sim g^{-3}$

[Epelbaum, FG, Jeon, Moore, Wu (2015)]



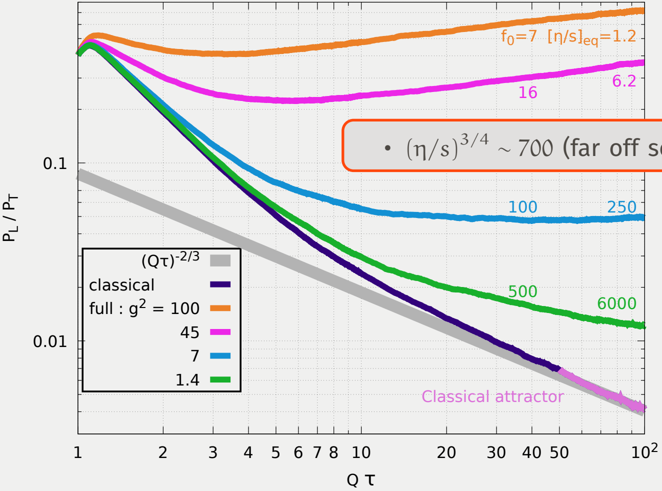
- With 1/2 : evolution depends on g^2
- Gives a different P_L/P_T
- Vertical bar indicates $(\eta/s)^{3/4} \sim g^{-3}$
- Agreement with classical result improves when g^2 decreases

[Epelbaum, FG, Jeon, Moore, Wu (2015)]



Isotropization in a longitudinally expanding system

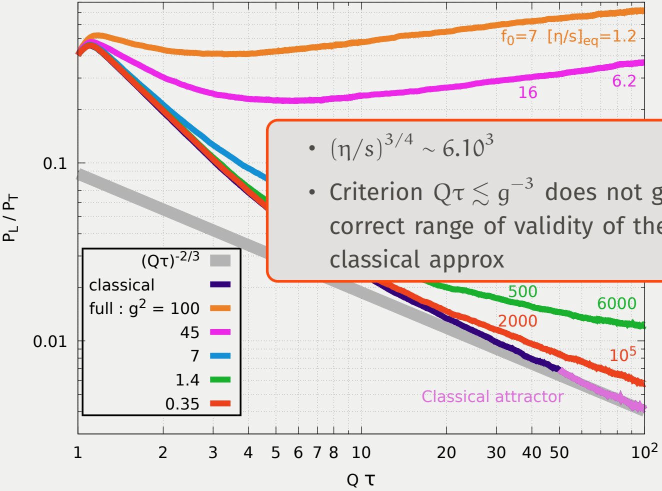
[Epelbaum, FG, Jeon, Moore, Wu (2015)]



- $(\eta/s)^{3/4} \sim 700$ (far off scale)

ISOTROPIZATION IN A LONGITUDINALLY EXPANDING SYSTEM

[Epelbaum, FG, Jeon, Moore, Wu (2015)]



- $(\eta/s)^{3/4} \sim 6.10^3$
- Criterion $Q\tau \lesssim g^{-3}$ does not give the correct range of validity of the classical approx

Why is the vacuum $1/2$ important ?

- The $1/2$'s ensure that the terms f^3 and f^2 are correct
- The quadratic terms are important in anisotropic systems

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- No 1/2 $\Longrightarrow$ no f^2 terms in Boltzmann eq. :

$$\partial_t f_4 \sim g^4 \int_{123} \cdots \left[f_1 f_2 (f_3 + f_4) - f_3 f_4 (f_1 + f_2)\right] + \cdots \left[\cancel{f_1 f_2 - f_3 f_4}\right]$$

WHY IS THE VACUUM $1/2$ IMPORTANT ?

- The $1/2$'s ensure that the terms f^3 and f^2 are correct
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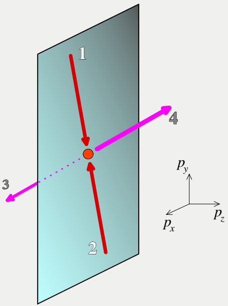

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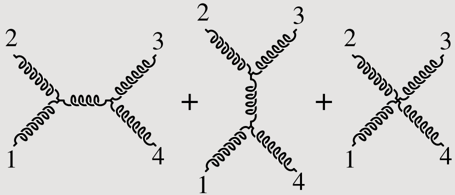
- When the distribution is very anisotropic, trying to produce the particle 4 at large angle results in $f_3 \approx f_4 \approx 0 \quad \Rightarrow$ nothing left
- Cubic terms $\Leftrightarrow$ stimulated emission : ineffective to produce particles in empty regions of phase-space

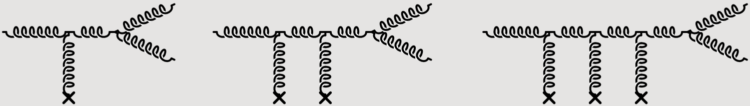
More insights from kinetic theory

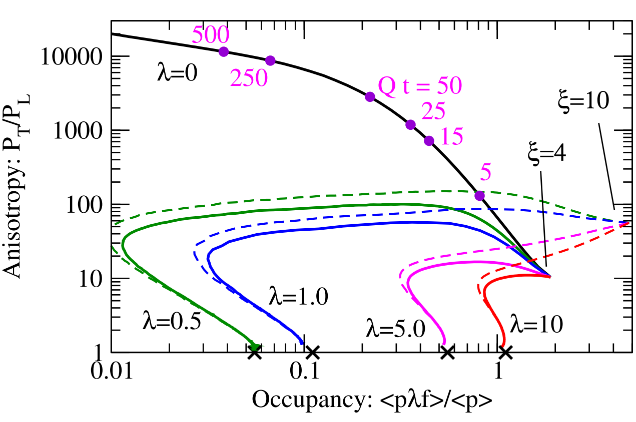
KINETIC THEORY FOR GLUONS [Kurkela, Zhu (2015)]

$2 \to 2$



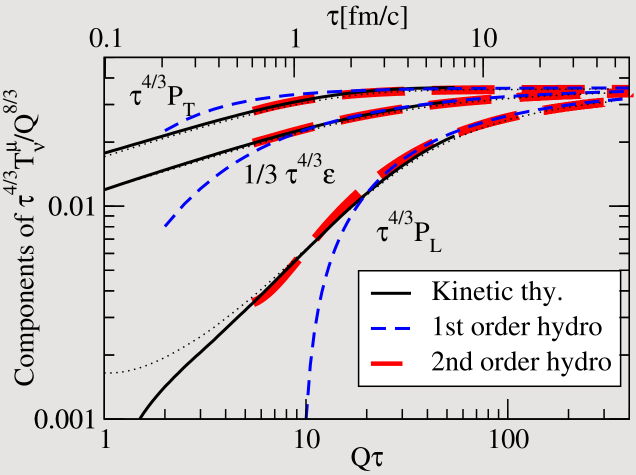
$1 \to 2$, $2 \to 1$ + Landau-Pomeranchuk-Migdal resummation





For $\lambda = 0.5$, the **Strong CSA** breaks down at $Q\tau \approx 2$, while simple estimates suggested that it would be valid up to $Q\tau \approx \alpha_s^{-3/2} \approx 350$

Consistent with hydrodynamics before full isotropization



Boltzmann in the relaxation time approcimation

$$\left(\partial_\tau - \frac{p_z}{\tau}\right) f(\tau, \boldsymbol{p}) = -\frac{f - f_{\mathrm{eq}}}{\tau_{\mathrm{R}}}$$

$\tau_{\mathrm{R}} \equiv$ relaxation time

$f_{\mathrm{eq}} \equiv$ local equilibrium dist

- $\tau_{\mathrm{R}} = \infty :$ no collisions
- $\tau_{\mathrm{R}} \sim \epsilon^{-1/4} :$ conformal; rate scales as inverse temperature
- $\tau_{\mathrm{R}} = \mathrm{const} :$ fixed collision rate (not realistic with expansion)

- Define moments :

$$L_n \equiv \int_{\mathbf{p}} \mathbf{p}^2 \; P_{2n}(p_z/p)\; f(\tau,\mathbf{p}) \quad , \quad g_n \equiv \tau\partial_\tau \ln L_n$$

$$L_0 = \epsilon = P_{_L} + 2P_{_T}, \quad L_1 = P_{_L} - P_{_T}$$

Boltzmann $\Leftrightarrow$ coupled equations for L_n

$$\begin{aligned}
\partial_\tau L_0 &= -\frac{a_o L_0 + c_0 L_1}{\tau} \\
\partial_\tau L_n &= -\frac{a_n L_n + c_n L_{n+1} + b_n L_{n-1}}{\tau} - \frac{L_n}{\tau_{_R}} \qquad (n \geq 1)
\end{aligned}$$

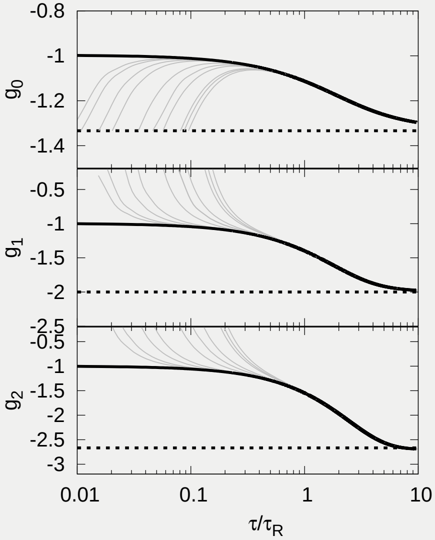
a_n, b_n, c_n = pure numbers, known explicitly (depend only on the free streaming part of Boltzmann eq.)

Free streaming fixed point ($\tau_R = \infty$)

- All the g_n behave as τ^{-1}, with fixed ratios
- $L_1/L_0 \to -\frac{1}{2}$, i.e. $P_L/P_T \to 0$

Interacting fixed point ($\tau_R \sim \epsilon^{-1/4}$)

- $g_0 \to -4/3, \quad g_1 \to -2$
- Locally isotropic distribution



- Universal attractor
- $\tau \lesssim \tau_R$: trajectories first approach free streaming fixed point
- $\tau \gtrsim \tau_R$: trajectories go to the local equilibrium fixed point

Two-Particle Irreducible framework

Is something missing ?

	Weak CSA	Strong CSA	Kinetic th.
Ultraviolet	✗	✓	✓
f^2 terms	✓	✗	✓

Is something missing ?

	Weak CSA	Strong CSA	Kinetic th.
Ultraviolet	✗	✓	✓
f^2 terms	✓	✗	✓
Screening	✓	✓	✗

Is something missing ?

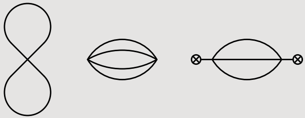
	Weak CSA	**Strong CSA**	**Kinetic th.**	**2-PI**
Ultraviolet	✗	✓	✓	✓
f^2 terms	✓	✗	✓	✓
Screening	✓	✓	✗	✓

Quantum effective action

$$\Gamma[\varphi, G] = S[\varphi] - \frac{i}{2}\mathrm{tr}\,\log G + \frac{i}{2}\mathrm{tr}\Big(\big(G_0^{-1} - G^{-1}\big)G\Big) + \Phi[\varphi, G]$$

$\Phi[\varphi, G] =$ sum of vacuum 2PI graphs

Truncation at order g^4

$\Phi[\varphi, G] =$ 

Equations of motion

$$\frac{\delta\Gamma}{\delta\varphi_x} = 0$$

$$\frac{\delta\Gamma}{\delta G_{xy}} = 0$$

Equations of motion

$$\sqrt{-g_x}\,\Big\{\nabla_\mu\nabla^\mu\varphi_x + V'(\varphi_x) + V'''(\varphi_x)G_{xx}\Big\} = \frac{\delta\Phi}{\delta\varphi_x}$$

$$\Big(\nabla^x_\mu\nabla^\mu_x + V''(\varphi_x)\Big)G_{xy} = -i\frac{1}{\sqrt{-g_x}}\delta(x-y)$$
$$-\int d^4z\,\sqrt{-g_z}\;\underbrace{-2\frac{1}{\sqrt{-g_x}}\frac{\delta\Phi}{\delta G_{xz}}\frac{1}{\sqrt{-g_z}}}_{\Sigma_{xz}}\,G_{zy}$$

Computational cost per time step

Kinetic theory (deterministic algorithm)

3d-isotropic : N^3

Long. expansion + azimuthal symm. : $\mathrm{N}^3\mathrm{N}_z^3$

2PI with $\mathrm{N}_{\mathrm{mem}}$-deep time memory integral

3d-isotropic : $\mathrm{N}\log(\mathrm{N}) \times \mathrm{N}_{\mathrm{mem}}^2$

Long. expansion + azimuthal symm. : $\mathrm{N}\mathrm{N}_z\log(\mathrm{N}\mathrm{N}_z) \times \mathrm{N}_{\mathrm{mem}}^2$

[So far, one implementation : **Hatta, Nishiyama (2013)**]

Summary

- LO : no pressure isotropization, NLO : instabilities
- Beyond NLO : **Classical statistical approximation**
  - Weak CSA :
    non-renormalizable, sensitive to UV cutoff
  - Strong CSA :
    underestimates large angle scatterings
    breaks rapidly unless η/s very large
- **Kinetic theory :** avoids all these difficulties, but does not cope well with screening effects in the soft region
- **Two-PI for longitudinally expanding systems :**
  Important to properly treat screening effects

Energy-momentum tensor

$$
\begin{aligned}
T^{\mu\nu} \quad = \quad & \nabla^\mu\varphi\nabla^\nu\varphi - g^{\mu\nu}\mathcal{L} + \Big[\nabla_x^\mu\nabla_y^\nu G_{xy}\Big]_{x=y} \\
& +\frac{1}{2}g^{\mu\nu}\Big\{V''(\varphi_x)G_{xx} - \Big[\nabla_\alpha^x\nabla_y^\alpha G_{xy}\Big]_{x=y}\Big\} \\
& -g^{\mu\nu}\frac{\delta\Phi}{\delta\sqrt{-g}}
\end{aligned}
$$

Isotropization in a fixed box