Asymptotic behaviour of the late-time expansion of supersymmetric Yang-Mills plasma

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State of the art

- Hydrodynamics works way better than it should ...
- Hydrodynamisation: damping of the the *transient* degrees of freedom and leaving long-lived *hydrodynamic* modes
- It is illustrated in a number of model computations
- The ultimate question:

  What is the meaning of the hydrodynamic expansion?

The setup

- Study late time dynamics of $\mathcal{N} = 4$ SYM via holographic methods
- Use Bjorken symmetry to constrain system's dynamics
- Consistent description via *resurgent transseries* expansion
- Hydrodynamics series encodes *all* transient modes and their couplings
- Transient degrees of freedom reflect the spectrum of quasinormal modes of the *AdS* black hole

Bjorken flow

- Boost invariant metric $ds^2 = -d\tau^2 + \tau^2 dy^2 + dx_\perp^2$
- Energy momentum tensor is diagonal

$$T_{\mu\nu} = \mathrm{diag}\{\epsilon(\tau), P_L(\tau), P_T(\tau), P_T(\tau)\}$$

- Conditions: $\nabla_\mu T^{\mu\nu} = 0$ and $T^\mu_\mu = 0$ imply

$$P_L = -\epsilon - \tau\dot{\epsilon}\ , \quad P_T = \epsilon + \frac{1}{2}\tau\dot{\epsilon}$$

- Evolution of the system is captured by a single function $\epsilon(\tau)$
- Strict for an infinite energy collision of infinitely large nuclei

J. D. Bjorken, Phys. Rev. D **27**, 140 (1983)

R. A. Janik, Lect. Notes Phys. **828**, 147 (2011)

Bjorken flow

- Energy density *defines* local effective temperature

$$\epsilon(\tau) = \frac{3}{8} N_c^2 \pi^2 \, T_{\mathrm{eff}}(\tau)^4$$

- Universal, dimensionless variable

$$w = \tau \, T_{\mathrm{eff}}(\tau)$$

J. D. Bjorken, Phys. Rev. D **27**, 140 (1983)
R. A. Janik, Lect. Notes Phys. **828**, 147 (2011)

- Eddington-Finkelstein coordinates for large-τ expansion

$$ds^2 = -r^2 A d\tau^2 + 2 d\tau dr + (r\tau + 1)^2 e^b dy^2 + r^2 e^{-\frac{1}{2}b+d} dx_\perp^2$$

- Einstein's equations $R_{ab} + 4g_{ab} = 0$ with $\Lambda = -6$
- Evolution equations are first order in time

S. Kinoshita, *et al.* Prog. Theor. Phys. **121**, 121 (2009)
P. M. Chesler and L. G. Yaffe, JHEP **1407**, 086 (2014)

The scaling variable

- Motivated by the naive localisation of the event horizon fix

$$s = \frac{1}{r}\tau^{-\frac{1}{3}}$$

with $0 \leq s \leq 1$

- Gradient corrections are quantified by $\frac{1}{\tau T} \sim \tau^{-\frac{2}{3}}$ so

$$u = \tau^{\frac{2}{3}}$$

R. A. Janik, R. B. Peschanski, Phys. Rev. D **73**, 045013 (2006)

Large-τ expansion

$$A(r,\tau) = \sum_{\boldsymbol{n}\in\mathbb{N}_{\boldsymbol{0}}^{\infty}} \Omega_{\boldsymbol{n}}(u) \sum_{i=0}^{\infty} u^{-i} A_i^{(\boldsymbol{n})}(s)$$

$$b(r,\tau) = \sum_{\boldsymbol{n}\in\mathbb{N}_{\boldsymbol{0}}^{\infty}} \Omega_{\boldsymbol{n}}(u) \sum_{i=0}^{\infty} u^{-i} b_i^{(\boldsymbol{n})}(s)$$

$$d(r,\tau) = \sum_{\boldsymbol{n}\in\mathbb{N}_{\boldsymbol{0}}^{\infty}} \Omega_{\boldsymbol{n}}(u) \sum_{i=0}^{\infty} u^{-i} d_i^{(\boldsymbol{n})}(s)$$

$$\Omega_{\boldsymbol{n}}(u) = u^{-\boldsymbol{\alpha}_{\boldsymbol{n}}}\, \mathrm{e}^{-\boldsymbol{n}\cdot\boldsymbol{A}\, u} \qquad \boldsymbol{A} = \left(A_1, \overline{A_1}, A_2, \overline{A_2}, \cdots\right)$$

Hydrodynamic sector

- $\mathbf{n = 0}$ case is called hydrodynamic sector with the 0th order solution

$$d_0^{(\mathbf{0})}(s) = 0, \quad A_0^{(\mathbf{0})}(s) = 1 - s^4, \quad b_0^{(\mathbf{0})}(s) = 0$$

- To find $d_j^{(\mathbf{0})}(s)$, $A_j^{(\mathbf{0})}(s)$, $b_j^{(\mathbf{0})}(s)$ for $j > 0$ one solves a hierarchy of coupled ODEs
- Using holographic renormalization one gets the desired energy density

M. P. Heller *et al.* Phys. Rev. Lett. **110**, no. 21, 211602 (2013)

J. Casalderrey-Solana *et al.* JHEP **1804**, 042 (2018)

Leading transient sector

- The modes obey

$$d_0^{(\boldsymbol{e}_1)}(s) = 0, \qquad A_0^{(\boldsymbol{e}_1)}(s) = 0, \qquad b_0^{(\boldsymbol{e}_1)}(s) = Z_{\boldsymbol{e}_1}(s)$$

where $\mathbf{e}_k = (...0, \underbrace{1}_{2k-1}, 0...)$ and $Z_{\boldsymbol{e}_1}(s)$ satisfies

$$\left(s(1-s^4)\partial_s^2 - (3 + s^4 - 2i\, \boldsymbol{e}_1 \cdot \boldsymbol{\omega}\, s)\partial_s - 3i\, \boldsymbol{e}_1 \cdot \boldsymbol{\omega} \right) Z_{\boldsymbol{e}_1}(s) = 0$$

and $\boldsymbol{\omega} = -\frac{2i}{3}\boldsymbol{A}$ and $\boldsymbol{\alpha} = \frac{\boldsymbol{A}}{6}$

- An eigenvalue problem with infinitely many solutions
- Spectrum reflects QNMs of the *AdS* black hole
- We identify $Z_{\boldsymbol{e}_1}(s=1) = \boldsymbol{\sigma}$ with transseries parameters

For $i \geq 1$ and $\mathbf{n} = \mathbf{e}_1$ or $i \geq 0$ for $n > \mathbf{e}_1$ we have

$$\mathcal{L}^d_{\boldsymbol{n}} d^{(\boldsymbol{n})}_i = j^{d,\boldsymbol{n}}_i$$
$$\mathcal{L}^A_{\boldsymbol{n}} A^{(\boldsymbol{n})}_i = j^{A,\boldsymbol{n}}_i$$
$$\mathcal{L}^b_{\boldsymbol{n}} b^{(\boldsymbol{n})}_i = j^{b,\boldsymbol{n}}_i$$

with linear operators

$$\mathcal{L}^d_{\boldsymbol{n}} = \partial_s^2$$
$$\mathcal{L}^A_{\boldsymbol{n}} = s\partial_s - 4$$
$$\mathcal{L}^b_{\boldsymbol{n}} = s(1-s^4)\partial_s^2 - (3 + s^4 - 2i\,\boldsymbol{n}\cdot\boldsymbol{\omega}\,s)\partial_s - 3i\,\boldsymbol{n}\cdot\boldsymbol{\omega}$$

Transseries expansion of energy density

- The energy density admits an expansion

$$\mathcal{E}(u,\boldsymbol{\sigma}) = \sum_{\boldsymbol{n}\in\mathbb{N}_0^\infty} \boldsymbol{\sigma}^{\boldsymbol{n}}\, \mathrm{e}^{-\boldsymbol{n}\cdot\boldsymbol{A}\,u}\, \Phi^{(\boldsymbol{n})}(u)$$

where

$$\Phi^{(\boldsymbol{n})}(u) = u^{-\beta_{\boldsymbol{n}}} \sum_{k=0}^{+\infty} \varepsilon_k^{(\boldsymbol{n})}\, u^{-k}$$

- Using holographic renormalization

$$\varepsilon_i^{(\boldsymbol{n})} = -\frac{1}{\boldsymbol{\sigma}^{\boldsymbol{n}}}\frac{1}{4!}\frac{d^4}{ds^4}A_i^{(\boldsymbol{n})}(s=0)$$

- Each "instanton action" A_k comes with an independent transseries parameter σ_k

Sapmle sectors

- Hydro sector

$$\Phi^{(\mathbf{0})}(u) = u^{-\beta_{\mathbf{0}}} \sum_{k=0}^{\infty} \varepsilon_k^{(\mathbf{0})} u^{-k}$$

with $\beta_{\mathbf{0}} = 2$. We have first 380 coefficients

- The first transient sector

$$\Phi_{\boldsymbol{e}_1}(u) = u^{-\beta_{\boldsymbol{e}_1}} \sum_{k=0}^{\infty} \varepsilon_k^{(\boldsymbol{e}_1)} u^{-k}$$

$\beta_{\boldsymbol{e}_1} = -\mathrm{i}\,\frac{\omega_0}{4} + 3$ with $\omega_0 = 3.1195\cdots - \mathrm{i}\,2.7467\cdots$
We have first 250 coefficients

Higher modes and non-linear effects

- Higher transient mode (2nd QNM)

$$\Phi_{\boldsymbol{e}_2}(u) = u^{-\beta_{\boldsymbol{e}_2}} \sum_{k=0}^{\infty} \varepsilon_k^{(\boldsymbol{e}_2)} u^{-k}$$

where $\beta_{\boldsymbol{e}_2} = -\mathrm{i}\,\frac{\omega_1}{4} + 3$ where $\omega_1 = 5.16952\cdots - \mathrm{i}\,4.76357\cdots$
We have first 200 coefficients

- Multi-mode contributions

$$\Phi_{2\boldsymbol{e}_1}(u) = u^{-\beta_{2\boldsymbol{e}_1}} \sum_{k=0}^{\infty} \varepsilon_k^{(2\boldsymbol{e}_1)} u^{-k}$$

where $\beta_{2\boldsymbol{e}_1} = 2\beta_{\boldsymbol{e}_1} - 2$. We have first 100 coefficients

High order coefficients

- For $n \gg 1$

$$\varepsilon_n^{(0)} \sim \frac{\Gamma(n+\beta)}{A_1^{n+\beta}} + \mathrm{c.c.}$$

with $\beta = \beta_{\mathbf{0}} - \beta_{\mathbf{e}_1} \in \mathbb{C}$ and $A_1 \in \mathbb{C}$, $(\beta_{\mathbf{0}} = 2)$

- $A_1 = \mathrm{i}\frac{3}{2}\omega_0$ is the frequency of the transient QNM mode
- $\mathrm{Im}\ \omega_0 \sim \tau_0^{-1}$ where τ_0 is the equilibration time
- At RHIC and LHC $\tau_0 \sim 0.5 - 1\ \mathrm{fm/c}$
- All information is stored in the coefficients the hydrodynamic gradient expansion: resurgence property

Large order relations

- For the hydrodynamic series at the leading singularity

$$\varepsilon_n^{(0)} \sim -\frac{S_{0\to e_1}}{2\pi\mathrm{i}} \frac{\Gamma(n+\beta)}{A_1^{n+\beta}} \left(\varepsilon_0^{(e_1)} + \frac{A_1 \varepsilon_1^{(e_1)}}{n+\beta-1} + \right.$$

$$\left. + \frac{A_1^2 \varepsilon_2^{(e_1)}}{(n+\beta-1)(n+\beta-2)} + \cdots \right) + \mathrm{c.c.} + \cdots$$

  where $S_{0\to e_1}$ is the Stokes constant ($\beta = \beta_{\mathbf{0}} - \beta_{e_1}$)

- Large order relations contain contributions from *all* sectors and couplings between them

- Every sector has an independent Stokes constant $S_{0\to e_k}$

The Borel transform

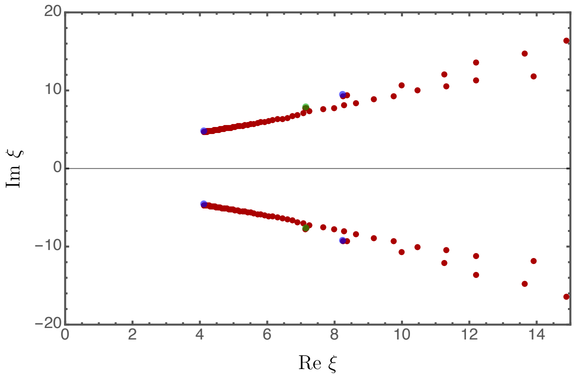
- The Borel transform is defined

$$\mathcal{B}\left[\Phi\right](\xi) = \sum_{n=0}^{\infty} \frac{F_n}{\Gamma(n+\beta)} \xi^{n+\beta-1}$$

if $F_n \sim \Gamma(n+\beta)$ and $\Phi(x) = \sum_{n=1}^{\infty} F_n x^{-n-\beta}$

- Due to finite number of terms we have to perform Borel-Padé approximant
- We analyse poles of the Borel-Padé approximant $\mathrm{BP}_N\left[\Phi\right](\xi)$ to estimate singularities of the Borel transform
- First square root branch points appear at $\xi = A_1$ and $\xi = \overline{A_1}$

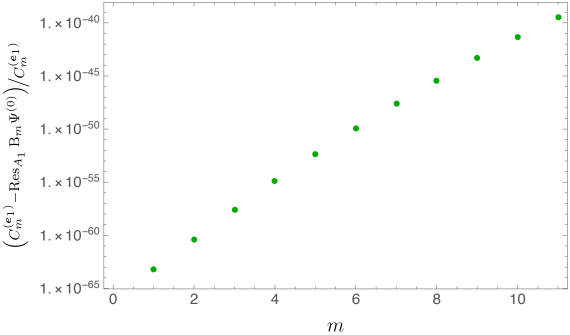
Leading hydrodynamic singularity



Poles of the Borel-Padé approximant $\mathrm{BP}_{189}\,[\epsilon_{\mathrm{hydro}}]$, in the complex ξ−plane $\quad \xi = A_1, \overline{A_1}, 2A_1, 2\overline{A_1} \qquad \xi = A_2, \overline{A_2}$

$$S_{0\to \boldsymbol{e_1}} = 0.01113\cdots - i0.03050\cdots$$

Leading hydrodynamic singularity



Comparison of the resurgence predictions of the Borel-Padé approximant (of order $N = 189$) of the hydrodynamic series, and the predicted value of the expansion around the $\xi = A_1$ singularity

Sub-leading hydrodynamic singularity

- Subtract the leading order contribution from the hydro coefficients

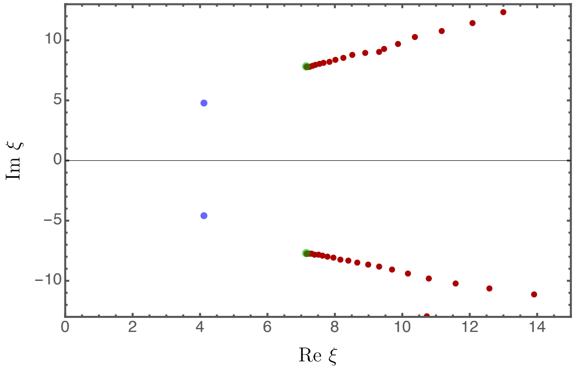
$$\delta_1 \varepsilon_n^{(0)} = \varepsilon_n^{(0)} + \frac{S_{0\to e_1}}{2\pi \mathrm{i}} \frac{\Gamma(n+\beta)}{A_1^{n+\beta}} \left(\varepsilon_0^{(e_1)} + \frac{A_1 \varepsilon_1^{(e_1)}}{n+\beta-1} + \right.$$

$$\left. + \frac{A_1^2 \varepsilon_2^{(e_1)}}{(n+\beta-1)(n+\beta-2)} + \cdots \right) + \mathrm{c.c.}$$

- The leading contribution to the $\delta_1 \varepsilon_n^{(0)}$ is determined by the modes A_2 and $\overline{A_2}$
- We can perform the same analysis for the series

$$\delta_1 \Phi_0(u) \simeq u^{-\beta_0} \sum_{n=1}^{+\infty} \delta_1 \varepsilon_n^{(0)} \, u^{-n} = u^{-\beta_0 - 1} \sum_{n=0}^{+\infty} \delta_1 \varepsilon_{n+1}^{(0)} \, u^{-n}$$

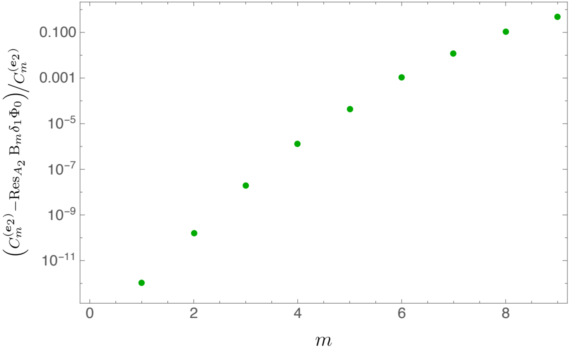
Sub-leading hydrodynamic singularity



Poles of the Borel-Padé approximant $\mathrm{BP}_{90}\left[\delta_1 \Phi_0\right]$

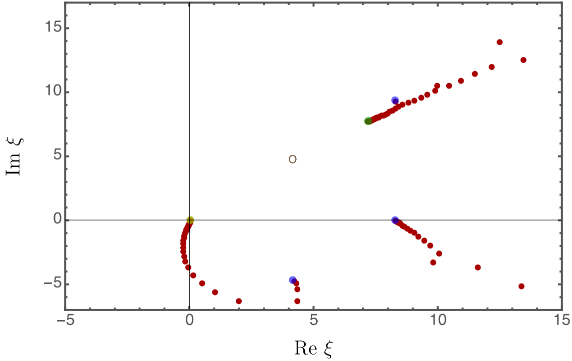
$$S_{0\to \boldsymbol{e}_2} = 0.0183\cdots - \mathrm{i}0.01161\cdots$$

Sub-leading hydrodynamic singularity



Comparison of the resurgence predictions of the Borel-Padé approximant (of order $N = 90$) of the corrected hydrodynamic series $\delta_1\Phi_0$, and the predicted value of the expansion around the $\xi = A_2$ singularity

Leading behaviour in the first non-hydro sector

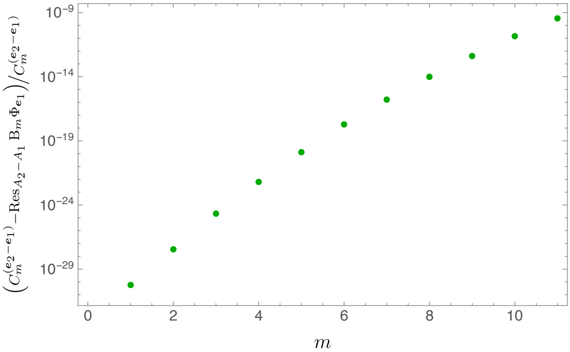


Poles of the Borel-Padé approximant $\mathrm{BP}_{135}\left[\Phi_{\boldsymbol{e}_1}\right]$

$\xi = \overline{A_1} - A_1, A_1, \overline{A_1}, \qquad \xi = A_2 - A_1$

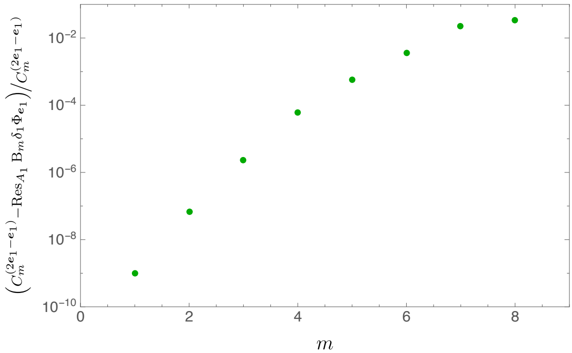
$$S_{\boldsymbol{e}_1 \to \boldsymbol{e}_2} = -0.91134 - \mathrm{i}\ 0.81107$$

Leading behaviour in the first non-hydro sector



Comparison of the resurgence predictions of the Borel-Padé approximant (of order $N = 135$) of the series $\Phi_{\boldsymbol{e}_1}$, and the predicted value of the expansion around the $\xi = A_2 - A_1$ singularity

Effects of the QNM couplings



Comparison of the resurgence predictions of the Borel-Padé approximant (of order $N = 135$) of the series $\Phi_{\boldsymbol{e}_1}$, and the predicted value of the expansion around the $\xi = A_2 - A_1$ singularity

$$S_{\boldsymbol{e}_1 \to 2\boldsymbol{e}_1} = 2S_{0\to\boldsymbol{e}_1}$$

Summary

- Numerical evidence for resurgence in ab-initio computed strongly coupled QFT
- Transeries antzats provides a unified formalism to incorporate hydro and transient modes and their coupling
- Scaling limit is generalized to the whole time evolution
- Initial conditions are encoded in the σ parameters
- Possible implications for attractor ...