# A defect action for Wilson loops

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Holography and Extreme Chromodynamics
Santiago de Compostela,
July 2, 2018

# Outline

- Introduction
- Effective action
- Circular Wilson loop
- Outlook

- A Wilson loop is the holonomy of the gauge field along a closed curve $\mathcal{C}$

$$\langle W_R(\mathcal{C}) \rangle = \left\langle \mathrm{Tr}\,_R \mathcal{P}\, e^{i \oint_{\mathcal{C}} A} \right\rangle$$

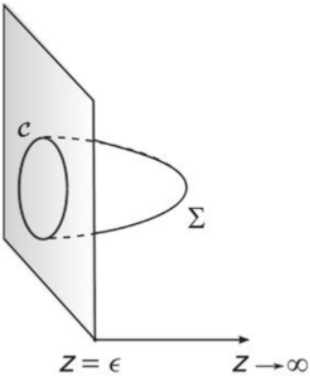
- This can be generalized to closely related operators e.g. $\mathcal{N} = 4$ 1/2-BPS loop

$$\langle W_{BPS}(\mathcal{C}) \rangle = \left\langle \mathrm{Tr}\,_R \mathcal{P} \, \exp \left( i \int ds \left( \dot{x}^\mu A_\mu + |\dot{x}| \phi^i \theta_i \right) \right) \right\rangle$$

- Wilson loops are among the most fundamental observables in Yang-Mills theory

# Holographic dual of a Wilson loop

- Two-dimensional surface $\sim AdS_2$ [Maldacena '98]



# Holographic dual of a Wilson loop

- Dual of Wilson loop is a dynamical string $\longrightarrow$ effective 1d theory? Connection to SYK models?

| | |
|---|---|
| SYK | 1d fermions $\to$ emergent 2d gravity |
| Wilson loop | 1d effective theory $\overset{?}{\to}$ 2d string |

- Could be useful for semi-holographic phenomenology:

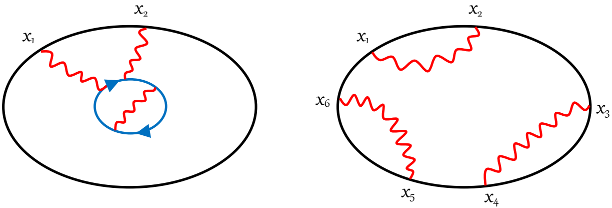
| | |
|---|---|
| confining IR | holographic dual (e.g. Witten QCD) |
| asymptotically free UV | 1d effective theory |

# Weak coupling calculation

A "brute-force" calculation of a Wilson loop can be quite involved

$$\langle W(\mathcal{C})\rangle = \mathrm{Tr}_R \left(1 + ig\int d\tau \dot{x}^\mu(\tau)\langle A_\mu[x(\tau)]\rangle\right.$$

$$\left. + i^2 g^2 \int d\tau_1 d\tau_2 \dot{x}^\mu(\tau_1)\dot{x}^\nu(\tau_2)\langle A_\mu[x(\tau_1)]A_\nu[x(\tau_2)]\rangle \Theta(\tau_1-\tau_2) + \cdots \right)$$




"Simple" and "complicated" diagrams at the same order

Is there an organizing principle?

- A Wilson loop could also be seen as a defect or "impurity" with charged fields localized on it
- This is the case for $\mathcal{N}=4$ SYM BPS loops, that can be mapped to D-brane intersections [Gomis, Passerini '06]
    - Antisymmetric: $D5$ wrapping $S^4 \subset S^5$
    - Symmetric: $D3$ wrapping $S^2 \subset AdS_5$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $D3$ | • | • | • | • | – | – | – | – | – | – |
| $D5$ | • | – | – | – | – | • | • | • | • | • |
| $D3'$ | • | • | • | • | $\infty$ | – | – | – | – | – |

- We can use actions of BPS loops directly as starting point
- Simplest case: fundamental representation of $(S)U(N)$
- Matter at the defect:

| | |
|---|---|
| fermion in fundamental representation | $\chi$ |
| Abelian $U(1)$ gauge field | $a_\tau$ |

- Action: $S_{\text{def}} = S_W + S_{CS}$

$$S_W = \int_0^1 d\tau\, \chi^\dagger\, i(\partial_\tau - i a_\tau - i A_\tau)\chi, \quad A_\tau = \dot{x}^\mu(\tau) A_\mu[x(\tau)]$$

$$S_{CS} = \int_0^1 d\tau\, k a_\tau, \quad k = -1$$

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- Action: $S_{\text{def}} = S_W + S_{CS}$

$$S_W = \int_0^1 d\tau \, \chi^\dagger \, i(\partial_\tau - ia_\tau - iA_\tau)\chi, \quad A_\tau = \dot{x}^\mu(\tau) A_\mu[x(\tau)]$$

$$S_{CS} = \int_0^1 d\tau \, \left( ka_\tau + \frac{1}{2} \text{tr}\, A_\tau \right), \quad k = \frac{N}{2} - 1$$

- Proposal for expectation value of the Wilson loop

$$\langle W(\mathcal{C})\rangle \overset{?}{=} \mathcal{N}\int \mathcal{D}A_\mu \mathcal{D}\Phi\mathcal{D}\chi\mathcal{D}\chi^\dagger\mathcal{D}a_\tau\, e^{iS_{YM}[A_\mu,\Phi]+iS_{\rm def}}$$

- Seems reasonable, but is it correct?

- Proposal for expectation value of the Wilson loop

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- Seems reasonable, but is it correct?
- Most easily checked using Lorenz gauge on defect

$$\partial_\tau a_\tau = 0, \quad \partial_\tau A_\tau = 0$$

- Global $SU(N)$ transformation $\chi \to U\chi$,

$$a_\tau = a_0 \mathbb{1}, \quad A_D = U^\dagger A_\tau U = \text{diag}\,(a_1, \cdots, a_N)$$

- Integrating out the fermions

$$\langle W(\mathcal{C})\rangle \stackrel{?}{=} \mathcal{N}\left\langle \int_0^{2\pi} da_0\, e^{iS_{CS}}\, \det\left(i\partial_\tau + a_0 + A_D\right)\right\rangle$$

Calculation of the determinant

- Expand in Fourier modes $\chi(\tau+1)=-\chi(\tau)$

$$\chi(\tau)=\sum_{n=-\infty}^{\infty} e^{-2\pi i\left(n+\frac{1}{2}\right)\tau}\chi_n$$

- Using $\zeta$-function regularization

$$\det\left(i\partial_\tau+a_\tau+A_\tau\right)=e^{-i\frac{N}{2}a_0-\frac{i}{2}\sum_{j=1}^{N}a_j}\prod_{i=1}^{N}\left(1+e^{ia_0+ia_i}\right)$$

Calculation of the determinant

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$$\det\left(i\partial_\tau + a_\tau + A_\tau\right) = \underbrace{e^{-i\frac{N}{2}a_0 - \frac{i}{2}\sum_{j=1}^{N} a_j}}_{\text{cancels with } S_{CS}} \prod_{i=1}^{N}\left(1+e^{ia_0+ia_i}\right)$$

$$\langle W(\mathcal{C})\rangle \stackrel{?}{=} \mathcal{N}\left\langle \int_0^{2\pi} da_0\, e^{-ia_0} \prod_{i=1}^{N}\left(1+e^{ia_0+ia_i}\right)\right\rangle$$

$$\langle W(\mathcal{C})\rangle = \mathcal{N}\left\langle \int_0^{2\pi} da_0\, e^{-ia_0} \prod_{i=1}^{N} \left(1 + e^{ia_0+ia_i}\right) \right\rangle = \mathcal{N}\left\langle \sum_{i=1}^{N} e^{ia_i} \right\rangle$$

- Right formula up to the normalization
- Some subtleties in the gauge fixing: $A_\mu$ is defined on the whole spacetime

# Constructing the effective action

- Rewrite the expectation value

$$\langle W(\mathcal{C})\rangle = \int_0^{2\pi} da_0 e^{ika_0} \mathcal{D}\chi^\dagger \mathcal{D}\chi e^{i\tilde{S}_W} \left\langle e^{i\int d^4x\, J^\mu \cdot A_\mu} \right\rangle$$

- Free defect action

$$\tilde{S}_W = \int_0^1 d\tau\, \chi^\dagger\, i\left(\partial_\tau - ia_0\right)\chi,$$

- $U(N)$ Current

$$J^{a\,\mu}(x) = \int_0^1 d\tau\, \dot{x}^\mu(\tau)\, j^a(\tau)\delta^{(4)}\left(x - x(\tau)\right)$$

$$\boxed{j^a(\tau) = \chi^\dagger T^a \chi + \frac{\sqrt{N}}{2\sqrt{2}}\delta^{a0}}$$

# Constructing the effective action

- Interaction terms in defect action:

$$\left\langle e^{i\int d^4x\, J^\mu\cdot A_\mu}\right\rangle = e^{iW[J]}$$

$W[J]$= Generating functional for external current $J$

- Renormalization scale $\mu$ such that $g \ll 1$
- Expansion in connected time-ordered correlators

$$iW[J] = \frac{i^2}{2}\int\int d^4x d^4y \left\langle T\left(A_\mu^a(x)A_\nu^b(y)\right)\right\rangle_c J^{a\,\mu}(x)J^{b\,\nu}(y) + \cdots$$

# Constructing the effective action

- Interaction terms in defect action: vertices of $U(N)$ defect currents

$$iW[J] = \sum_{n=2}^{\infty} iW_n = \sum_{n=2}^{\infty} \frac{i^n}{n!} \int \prod_{i=1}^{n} \left(d\tau_i j^{a_i}(\tau_i)\right) K^{(n)\, a_1 \cdots a_n}(\tau_1, \cdots, \tau_n)$$

- Kernels: pullback of time-ordered correlators

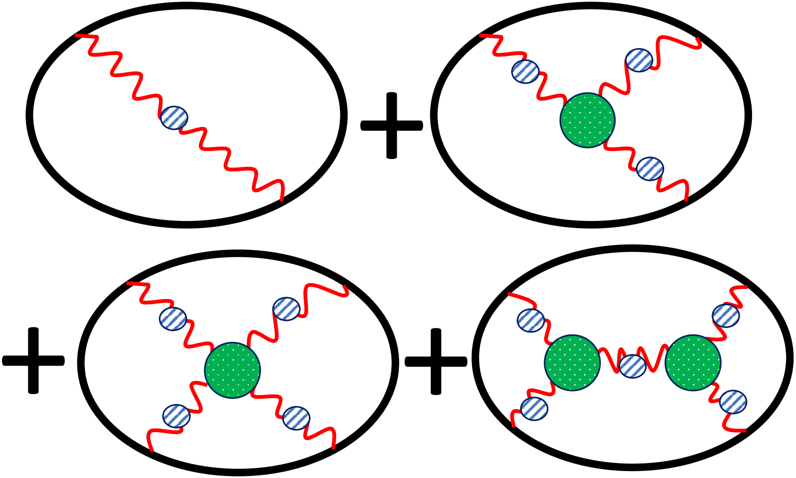
$$K^{(n)\, a_1 \cdots a_n}(\tau_1, \cdots, \tau_n) = \dot{x}(\tau_1)^{\mu_1} \cdots \dot{x}(\tau_n)^{\mu_n} G^{a_1 \cdots a_n}_{\mu_1 \cdots \mu_n}(x(\tau_1), \cdots, x(\tau_n))$$

- Time-ordered correlators

$$G^{a_1 \cdots a_n}_{\mu_1 \cdots \mu_n}(x_1, \cdots, x_n) = \left\langle T\left(A^{a_1}_{\mu_1}(x_1) \cdots A^{a_n}_{\mu_n}(x_n)\right)\right\rangle$$

# Constructing the effective action

- Time-ordered correlators can be computed from tree-level Feynman diagrams with 1PI propagators and vertices



# Leading order action

- To leading order, the 1PI action equals the classical action with renormalized couplings
- For a small Wilson loop $\mu = 1/L \Rightarrow$ weak coupling expansion valid for $g(1/L) \ll 1$
- Propagator in $R_\xi$ gauge

$$G_{\mu\nu}(x_{12}) = \underbrace{\frac{g^2}{8\pi^2|x_{12}|^2}\left[(1+\xi)\eta_{\mu\nu} - 2(\xi-1)\frac{(x_{12})_\mu (x_{12})_\nu}{|x_{12}|^2}\right]}_{\text{“classical”}} + O(g^4)$$

- Then, to $O(g^2)$ the effective action has a quartic fermion interaction

$$S_{\text{def}} = \int d\tau \chi^\dagger\, i(\partial_\tau - i a_0)\chi + \frac{i}{2}\int d\tau_1 d\tau_2\, j^{a_1}(\tau_1) j^{a_2}(\tau_2) K_{\text{cl}}^{(2)\, a_1 a_2}(\tau_1,\tau_2)$$

# Divergences

- The two-current kernel has a divergence linear in the UV cutoff $\Lambda$

$$S_{\rm def} \simeq (3-\xi)\Lambda \frac{ig^2}{4\pi^2}\int d\tau\, e j^a(\tau) j^a(\tau), \quad e = |\dot{x}|$$

- Can be removed by a local counterterm
- Divergence is absent in Yennie gauge $\xi = 3$
- Analogous to BPS loops: divergence absent in Feynman gauge $\xi = 1$ due to cancellations between gauge fields and scalars

  [Drukker, Gross, Ooguri '99,Erickson, Semenoff, Zarembo '00]

- Wilson loops have only linear divergences (once the couplings are renormalized)

$$\langle W(\mathcal{C})\rangle \sim e^{-\Lambda L}$$

  [Polyakov '80; Gervais, Neveu '80; Dotsenko, Vergeles '80]

- Effective action finite to all orders in Yennie gauge (?)

# Divergences

- Divergences appear when endpoints of propagators at the Wilson loop become coincident
- One can argue that the leading divergence (all points coincident) always vanishes
  - Bosonic symmetry
  - Antisymmetric color structure

  } Antisymmetric spacetime structure
- Kernel at coincident points

$$K^{(n)} = \underbrace{\dot{x}^{\mu_1} \cdots \dot{x}^{\mu_1}}_{\text{symmetric}} \underbrace{G_{\mu_1 \cdots \mu_n}}_{\text{antisymmetric}} \to 0$$

- Explicit check: subleading (log) divergences cancel at $O(g^4)$

# What makes Yennie gauge special?

- Yennie gauge characterized for its **IR finiteness**: In QED wavefunction renormalization is IR finite
- Why does it remove UV divergence in Wilson loop?
  The Yennie propagator is transvere to the separation vector

$$x_{12}^{\mu} G_{\mu\nu}^{\xi=3}(x_{12}) = 0$$

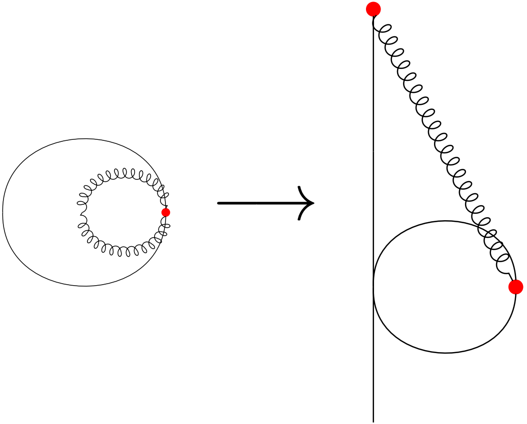
This implies covariance under inversion

$$x_{12}^{\mu} \longrightarrow \frac{x_{12}^{\mu}}{(x_{12})^2}$$

$$G_{\mu\nu}^{\xi} \longrightarrow \underbrace{(x_{12})^2 G_{\mu\nu}^{\xi=3}}_{\text{Yennie gauge propagator}} + \underbrace{\frac{\xi - 3}{8\pi^2}\eta_{\mu\nu}}_{\text{anomalous}}$$

# What makes Yennie gauge special?

Short-distance divergence $\longrightarrow$ IR divergence



Finite in Yennie gauge

# Circular Wilson loop

- Spatial Wilson loop defined on a circle of radius $R$

$$x^\mu = (0, R\cos(2\pi\tau), R\sin(2\pi\tau), 0)$$

- Two-current kernel in Yennie gauge

$$K^{(2)\,a_1 a_2} = -\frac{g^2}{2}\delta^{a_1 a_2}$$

- Four-fermion interaction factorizes

$$iW_2 = \frac{g^2}{4}\left(\int d\tau j^a(\tau)\right)^2$$

The interaction becomes “topological”

# Circular Wilson loop

- Write the interaction in terms of a Hermitian matrix integral

$$e^{iW_2} = \int [d\Sigma]\delta\left[\Sigma - \mathcal{O}\right] e^{-\frac{1}{4}(\mathrm{tr}\,(T^a\Sigma))^2}$$

$$\mathcal{O}_{ij} = (ig)\int d\tau\left(\chi_i^\dagger \chi_j + \frac{1}{2}\delta_{ij}\right)$$

(This is for $U(N)$, generalization to $SU(N)$ is straightforward)

- Introduce a second Hermitian matrix for the delta

$$\delta\left[\Sigma - \mathcal{O}\right] = \int [dM] e^{-iM_{ij}(\Sigma_{ij} - \mathcal{O}_{ij})}$$

- Integral for $\Sigma$ is Gaussian and can be done trivially

$$e^{iW_2} = \int [dM] e^{-g\int d\tau \chi^\dagger M \chi - 2\mathrm{tr}\,M^2 - \frac{g}{2}\mathrm{tr}\,M}$$

# Circular Wilson loop

- Integrating out the fermions

$$\langle W(R)\rangle = \mathcal{N}\int_0^{2\pi} da_0 e^{ika_0}\int [dM]\,\det(i\partial_\tau + a_0 + igM)e^{-2\mathrm{tr}\,M^2 - \frac{g}{2}\mathrm{tr}\,M}$$

- Evaluating the determinant and doing the integral over $a_0$, $M = \mathrm{diag}\,(M_1, \cdots, M_N)$

$$\langle W(R)\rangle = \mathcal{N}\int\left(\prod_{i=1}^{N} dM_i\right)\Delta^2(M)\left(\sum_{i=1}^{N} e^{-gM_i}\right)e^{-2\sum_{j=1}^{N} M_j^2}$$

$$\Delta^2(M) = \prod_{i<j}(M_i - M_j)^2$$

- Same as the exact result of $\mathcal{N} = 4$ SYM 1/2-BPS loop!

[Erickson, Semenoff, Zarembo '00; Drukker, Gross '00; Pestun '07]

# Outlook

- Map to holographic 2d string description
- NLO corrections: vertices and renormalization effects
- Effective theories for loops in different representations
- Large-$N$ analysis
- Unbounded curves: straight line and accelerated particle
- Cusps: anomalus dimensions, connection with scattering amplitudes