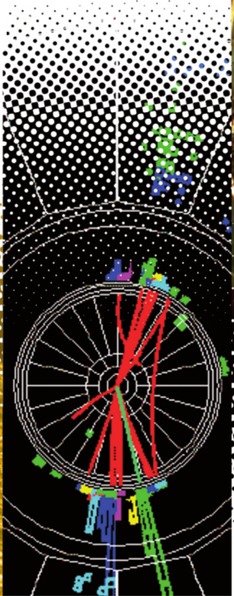
THE ELECTROWEAK UNIFICATION THEORY

Course in nine lectures

Bernardo Adeva Andany

Universidade de Santiago de Compostela



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INTRODUCTION

Supernovae explosion, combustion of solar hydrogen to form helium, heavy quark decay, or nuclear beta radiation, all weak interaction phenomena, are not unrelated to electromagnetism, but closely linked to it through the Higgs field.

This eBook contains a modern introduction to the electroweak unification theory, as part of the so called Standard Model of particle physics. Not only some of the key theoretical ideas are exposed in a precise way, but also the experiments that revealed them. The main highlights of the theory consolidation process are examined, the experimental counterpart of which spans over 40 years, from the discovery of neutral currents in 1973 to the Higgs boson in 2012.

The reader is assumed to have been introduced to quantum mechanics and quantum fields based on the local gauge invariance principle, and to be familiar with Dirac's relativistic electron theory. The course is specially suited for undergraduate students in physics, as part of an optional subject of elementary particles.

The course consists of nine lectures, that on the blackboard take about 90 minutes each. It contains a very select collection of problems and exercises, having as a connecting thread the calculation of the lifetime of elementary fermions and bosons, as well as the comprehension of some experimental results of historical relevance.

I am grateful to all those who contributed with their comments to improve the course, particularly to all undergraduates in Santiago that in late years have attended my classes, relentlessly spotting every typo or poor explanation. And I owe a debt of gratitude to Enrique Fernández Sánchez, for his priceless comments on specific physics topics.

Santiago de Compostela, February 12, 2024

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PARITY VIOLATION

DIRAC EQUATION AND CHIRALITY

Dirac equation: $(\not{p} - m)\psi = 0$

Free solutions:

$$\psi(x) = u(p,s)e^{-ipx},\ v(p,s)e^{ipx}$$

$\boldsymbol{p} = (0,0,p)$ (otherwise $\frac{p}{E+m} \rightarrow \frac{\boldsymbol{\sigma p}}{E+m}$):

$$u^{\uparrow} = N \begin{pmatrix} \uparrow \\ \frac{p}{E+m} \uparrow \end{pmatrix} \quad u^{\downarrow} = N \begin{pmatrix} \downarrow \\ \frac{-p}{E+m} \downarrow \end{pmatrix}$$

For positron, antiquark ($\boldsymbol{p} = (0,0,p)$):

$$v^{\uparrow} = N \begin{pmatrix} \frac{-p}{E+m} \downarrow \\ \downarrow \end{pmatrix} \quad v^{\downarrow} = N \begin{pmatrix} \frac{p}{E+m} \uparrow \\ \uparrow \end{pmatrix}$$

with $N = \sqrt{E+m}$

Reminder:

$$\not{p} \equiv p_\mu \gamma^\mu$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \equiv 2g^{\mu\nu}$$

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

Dirac-Pauli representation:

$$\gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \gamma^\mu = \begin{pmatrix} \gamma^0 & \vec{\gamma} \end{pmatrix}$$

A marvel of Dirac's theory, the ADJOINT spinor: $\bar{\psi} \equiv \psi^{*T}\gamma^0$
that allows creating relativistic quantities: $\bar{\psi}\Gamma_i\psi$
with $\Gamma_i = 1, \gamma^5, \gamma^\mu, \gamma^5\gamma^\mu, \sigma^{\mu\nu} = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu$
$i = \mathrm{scalar, pseudoscalar, vector, axial{-}vector, tensor}$

THE MEANING OF γ^5

- The operator γ^5 is called CHIRALITY
- Let us denote $k \equiv \frac{p}{E+m}$. The ultrarelativistic limits: $v \to c$, $m \to 0$, $p \gg m$, or $k \to 1$ are all equivalent, and it is customary to refer to them as *chiral limit*
- The helicity eigenstates $u^{\uparrow,\downarrow}$ are not eigenstates of γ^5
- But at high velocities, in the chiral limit, $u^{\uparrow,\downarrow}$ *approach* γ^5 eigenstates, and the γ^5 operator represents, in that limit, the helicity $\mathcal{H} \equiv 2\boldsymbol{S} \cdot \boldsymbol{p}/p$, having equal eigenvalues: $(+1, -1)$

Indeed: $\gamma^5 u^{\downarrow} = \gamma^5 \begin{pmatrix} \downarrow \\ -k\downarrow \end{pmatrix} = \begin{pmatrix} -k\downarrow \\ \downarrow \end{pmatrix} \xrightarrow{v\to c} \begin{pmatrix} -\downarrow \\ \downarrow \end{pmatrix} = -u^{\downarrow}$, eigenvalue: -1

and analogously $\gamma^5 u^{\uparrow} = u^{\uparrow}$, eigenvalue: $+1$

THE CHIRAL OPERATORS $\frac{1}{2}(1 \pm \gamma^5)$

- Each of the chiral operators $\gamma_{R,L} \equiv \frac{1}{2}(1 \pm \gamma^5)$ acts on BOTH helicities (to be called *equal* or *opposite* to the operator, in the following). For instance γ_L:

$$\frac{1}{2}(1-\gamma^5)u^\uparrow = \frac{1}{2}\begin{pmatrix} \uparrow -k\uparrow \\ -\uparrow +k\uparrow \end{pmatrix} = \frac{1}{2}(1-k)\begin{pmatrix} \uparrow \\ -\uparrow \end{pmatrix}$$

$$\frac{1}{2}(1-\gamma^5)u^\downarrow = \frac{1}{2}\begin{pmatrix} \downarrow +k\downarrow \\ -\downarrow -k\downarrow \end{pmatrix} = \frac{1}{2}(1+k)\begin{pmatrix} \downarrow \\ -\downarrow \end{pmatrix}$$

- Hence we can evaluate their action on any other quantum state of the fermion. In particular, on the state $u = \frac{1}{\sqrt{2}}u^\uparrow + \frac{1}{\sqrt{2}}u^\downarrow$ with zero average helicity $\langle\mathcal{H}\rangle = 0$:

$$\frac{1}{2}(1-\gamma^5)u = \frac{1}{2\sqrt{2}}\left[(1-k)\begin{pmatrix} \uparrow \\ -\uparrow \end{pmatrix} + (1+k)\begin{pmatrix} \downarrow \\ -\downarrow \end{pmatrix}\right]$$

whose helicity is now:

$$\langle\mathcal{H}\rangle_L = \frac{(1-k)^2-(1+k)^2}{(1-k)^2+(1+k)^2} = \frac{-p}{E} = -\beta = -\frac{v}{c}$$

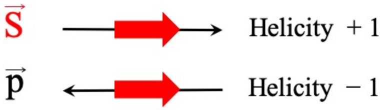
- The above expression is of great importance, for it provides the *exact polarization* induced by a chiral theory (γ_L) on the coupled fermion.

THE PARITY TRANSFORMATION

The PARITY transformation (P) consists in Quantum Mechanics in inverting the spatial coordinates, preserving the time arrow (thus the rotation $\vec{r} \times \vec{p}$ *sign*):

$$\vec{r} \to -\vec{r} \qquad \vec{p} \to -\vec{p} \qquad \vec{S} \to \vec{S}$$

Hence, P changes the fermion HELICITY:



- Therefore *every reaction that preferentially produces a given helicity of the fermions violates P*. According to today's level of knowledge, parity is strictly conserved by the strong interaction (QCD), by Maxwell's electromagnetic interaction (QED), both quantum and relativistic theories, and there is yet insufficient data concerning its possible violation in various approaches to quantum gravity.
- In 1956 T. D. Lee openly brought up the question: is parity violation possible in quantum physics?

DISCOVERY OF PARITY VIOLATION

In 1957, several experiments independently and unequivocally showed violation of parity in quantum processes of weak interactions. Two of them, published on January 15, did have historical importance:

- That of Wu, Ambler and Hudson at Washington DC, using cryogenics at mK temperatures to polarize the very high spin nucleus of $^{60}\mathrm{Co}$ (J=5), in the decay $^{60}\mathrm{Co} \to {}^{60}\mathrm{Ni}^* + e^- + \bar{\nu}_e$.
- That of Garwin, Lederman and Weinrich at Columbia University New York, utilising muons ($m_\mu = 105.7\ \mathrm{MeV}/c^2$), spontaneously polarized from pion decay (as we shall see), and properly stopped in graphite, in the decay: $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$.

In both cases, the electron yield with helicity -1 was significantly higher than that with helicity $+1$. Even if in both cases the initial state was polarized, the above is indeed a general fact.

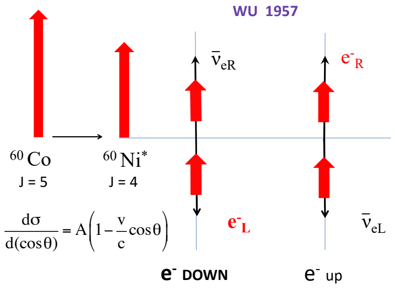
THE TWO KEY EXPERIMENTS

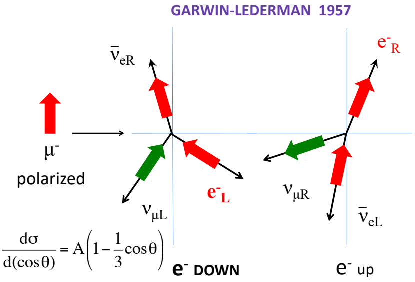
SIMILARITIES between both experiments:

- both used initial polarization, with no attempt to measure the outgoing e^- polarization
- both saw a negative asymmetry in electron emission, with respect to the direction of the initial spin: unequivocal sign of parity P violation
- in both cases the conservation of $\vec{J}$ and the negative asymmetry suggested a chiral solution with a left-handed electron and right-handed antineutrino $\bar{\nu}_e$
- in both cases the quantitative analysis revealed that the relativistic coupling is 100 % left-handed chiral: $\frac{1}{2}\gamma^\mu(1-\gamma^5)$

DIFFERENCES:

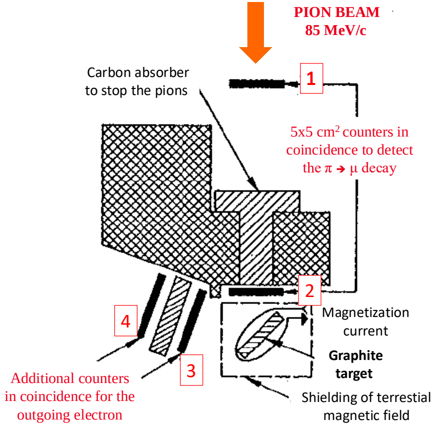
- While the $^{60}\mathrm{Co}$ is semirelativistic ($\gamma_{e^-} = 1.25$, with $v/c = 0.6$), the μ^- is ultrarelativistic ($\gamma_{e^-} = 220$)
- The photonic corrections are utterly important for $^{60}\mathrm{Co}$, but negligible for μ^-

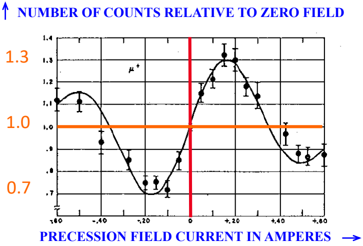




THE GARWIN-LEDERMAN EXPERIMENT

- The decay at rest $\pi^- \to \mu^- \bar{\nu}_\mu$ yields 100 % *polarized* muons.
- Larmor precession was induced to the μ^- at rest with frequency $\omega = g_\mu e|\vec{B}|/(2m_\mu)$
- The muon spin direction is kept unchanged, while it comes to rest in graphite.
- *The muon spin rotates (within the drawing's plane) one turn on average during its lifetime* $\tau_\mu = 2.20\,\mu$s.





The odd curve shows parity violation.

$1 + a\cos\theta$ was measured, with

$a = -0.33 \pm 0.03$

in good agreement with $1 - \frac{1}{3}\cos\theta$

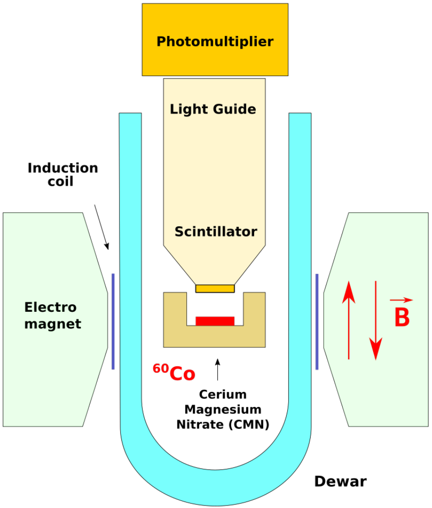
(see Problem 2)

$\theta = \omega t = \omega\tau_\mu$ ω=14KHz with $|\vec{B}|$=1G

- In addition $g_\mu = +2.00 \pm 0.10$ was measured for the muon, for the first time.

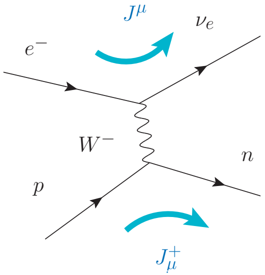
THE WU EXPERIMENT

- A single photomultiplier: $\cos\theta = +1$ (on top).
- More electrons are emitted in the direction opposite to the $\vec{B}$-field ($-$ 40 %).



- The high nuclear spin of ^{60}Co (J=5) together with the high spins of Mg and Ce in the CMN salt help nuclear polarization at mK temperatures.
- An electromagnet was used for adiabatic demagnetization *cooling*, and an induction coil was used to polarize the sample.
- The data show $a = -1$ in $1 + aP \cdot (v/c)\cos\theta$, with known $v/c = 0.6$ and $P = 0.67$ (polarization achieved in the ^{60}Co sample).

THE FERMI THEORY



the same $|\mathcal{M}|^2$ *for*
$\mathbf{p \to n e^+ \nu_e}$ and $\mathbf{p e^- \to n \nu_e}$
(e^- capture, supernovae)

$$\boldsymbol{n \to p\, e^- \bar{\nu}_e}$$

- In 1934 Enrico Fermi put forward a relativistic theory for nuclear β decay based on Dirac spinors and a vector coupling (γ^μ) between charged currents, inspired by electromagnetism, with amplitude:

$$\mathcal{M} = G_F(\bar{u}_n \gamma^\mu u_p)(\bar{u}_{\nu_e} \gamma_\mu u_e)$$

- The constant G_F is dimensionful $[\mathrm{GeV}^{-2}]$ [1] because it represents the propagator of a very massive spin 1 boson $g_w^2/(M_W^2 - q^2) \approx g_w^2/M_W^2$ in the limit $q^2 \ll M_W^2$, with $q^2 = (p_e - p_{\bar{\nu}_e})^2 = (p_n - p_p)^2$

- Fermi's theory had enormous success during the period 1934-1957 because it achieved a perfect understanding of all known β decay half-lifetimes and electron emission spectra, with a single coupling constant: $G_F = 1.16637 \times 10^{-5}\,\mathrm{GeV}^{-2}$ (current value).
- Given that $\gamma^\mu = \frac{1}{2}(1-\gamma^5)\gamma^\mu + \frac{1}{2}(1+\gamma^5)\gamma^\mu$, it becomes evident that such theory cannot interpret the observed parity violation, since it predicts equal numbers of right-handed electrons ($\mathcal{H} = +1$) and left-handed electrons ($\mathcal{H} = -1$).

[1] without adopting $\hbar = c = 1$, its exact dimension is energy $\times$ volume $[\mathrm{J\,m^3}]$

THE V−A THEORY

- In 1957 the V−A theory was established by Sudarshan, Marshak, Feynman, Gell-Man and Sakurai, which replaces Fermi's theory with the amplitudes:

$$\mathcal{M}(p \to n e^+ \nu_e) = \frac{4G_F}{\sqrt{2}} \left[\bar{u}_n \gamma^\mu \frac{1}{2}(1-\gamma^5) u_p \right] \left[\bar{u}_{\nu_e} \gamma_\mu \frac{1}{2}(1-\gamma^5) u_e \right]$$

$$\mathcal{M}(\mu^- \to e^- \bar{\nu}_e \nu_\mu) = \frac{4G_F}{\sqrt{2}} \left[\bar{\nu}_\mu \gamma^\mu \frac{1}{2}(1-\gamma^5) u_\mu \right] \left[\bar{u}_e \gamma_\mu \frac{1}{2}(1-\gamma^5) \nu_e \right]$$

- It conforms the general structure: $\mathcal{M} = \frac{4G_F}{\sqrt{2}} J^\mu J^+_\mu$. If J^μ "lowers the charge" , J^+_μ "raises the charge"(hermitic conjugate). Each one is left-handed chiral and may be formed by hadrons or leptons indistinctly.
- The V−A theory describes ALL weak interactions by charged currents in nuclear and particle physics. As of today, no violation is known to its strict *left-handed* character, being an essential part of the Standard Model.
- With the indicated $\sqrt{2}$ factor, G_F exactly recovers the same experimental values that it would have in Fermi's theory. With regard to the heavy boson mass, it obviously takes the value $G_F = \sqrt{2} g_w^2/(4M_W^2) = \sqrt{2} g^2/(8M_W^2)$ (units $\hbar = c = 1$). The dimensionless constant $g \equiv g_w \sqrt{2}$ will come up later when we define the Standard Model [2].

[2] the exact relationship is $G_F = \sqrt{2} g^2 (\hbar c)^3/[8(M_W c^2)^2]$ (units $\mathrm{J m^3}$), without adopting $\hbar = c = 1$.

β DECAY

LEFT-HANDED QUARKS AND LEPTONS

- Amplitudes $\frac{4G_F}{\sqrt{2}} J^\mu J_\mu^+$ can be formulated with *chiral left-handed* (called L or V−A) charged currents J^μ formed by *quarks or leptons* of *any two* of the following doublets, from the 3 known generations (21=$\binom{6}{2}+6$ forms):

$$\begin{pmatrix} u \\ d \end{pmatrix}_L \quad \begin{pmatrix} c \\ s \end{pmatrix}_L \quad \begin{pmatrix} t \\ b \end{pmatrix}_L \quad \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L$$

  given that the quark electric charge ($+\frac{2}{3}|e|$ UP type and $-\frac{1}{3}|e|$ DOWN type) raises or lowers by one unit $\Delta Q = \pm 1|e|$ at each doublet.

- These currents are analogous to the vector current $\gamma^\mu = \gamma^\mu(\gamma_R + \gamma_L)$ that couples quarks and leptons to the photon in QED (electric charge), or quarks to the gluon in QCD (color). Yet contrary to the above *only the left-handed chiral component* $\gamma^\mu \gamma_L$ is present.

- Note how, owing to $\overline{(1-\gamma^5)u_e} = u_e^{*T}(1-\gamma^5)\gamma^0 = u_e^{*T}\gamma^0(1+\gamma^5) = \bar{u}_e(1+\gamma^5)$, the chiral filter acts on BOTH fermions coupled to the current. For instance, with $J^\mu = \bar{u}_e \gamma^\mu \frac{1}{2}(1-\gamma^5)u_{\nu_e}$:

$$\bar{u}_e \gamma^\mu (1-\gamma^5) u_{\nu_e} = \bar{u}_e (1+\gamma^5)\gamma^\mu u_{\nu_e} = \overline{(1-\gamma^5)u_e} \;\; \gamma^\mu u_{\nu_e}$$

  that couples indistinctly to the (initial, final) state: (e_L^-, ν_L), $(\bar{\nu}_R, e_R^+)$, $(0, \nu_L e_R^+)$, $(e_L^- \bar{\nu}_R, 0)$, and analogously for $J^{\mu +}$.

THE CASIMIR RULE

- In Feynman diagram calculations, the following relations are useful, called *Casimir's rules*. The lemma $\sum_s \bar{u}(p)\Gamma u(p) = \mathrm{Tr}\left[(\not{p}+m)\Gamma\right]$ is easily shown from the basic property $\sum_s u\bar{u} = \not{p}+m$, and $[\bar{u}(a)\Gamma u(b)]^* = \bar{u}(b)\bar{\Gamma}u(a)$ is also straightforward [3]:

$$\sum_{\text{spins}} [\bar{u}(a)\Gamma_1 u(b)]\,[\bar{u}(a)\Gamma_2 u(b)]^* = \mathrm{Tr}\left[\Gamma_1(\not{p}_b + m_b)\bar{\Gamma}_2(\not{p}_a + m_a)\right]$$

$$\sum_{\text{spins}} [\bar{v}(a)\Gamma_1 v(b)]\,[\bar{v}(a)\Gamma_2 v(b)]^* = \mathrm{Tr}\left[\Gamma_1(\not{p}_b - m_b)\bar{\Gamma}_2(\not{p}_a - m_a)\right]$$

  where $\Gamma_i = 1, \gamma^5, \gamma^\mu, \gamma^5\gamma^u, \sigma^{\mu\nu}$ represent *any product* of γ matrices. We define $\bar{\Gamma} \equiv \gamma^0\Gamma^+\gamma^0$, with Γ^+ meaning conjugate and transposed. It can easily be shown that $\overline{\gamma_a\gamma_b\gamma_c\gamma_d} = \gamma_d\gamma_c\gamma_b\gamma_a$, but $\overline{\gamma^\mu(1-\gamma^5)} = \gamma^\mu(1-\gamma^5)$, and $\overline{1-\gamma^5} = 1+\gamma^5$.
- With $\Gamma_1 = \Gamma_2$ the above rules allow calculating $\sum_{ss'} |\mathcal{M}|^2$, but they also allow to handle interferences of the type $\sum_{ss'} \mathcal{M}_1{}^*\mathcal{M}_2$.
- Recall $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, and the cyclic property of the traces, that lets these rules be written in different ways.
- For the mixed cases we have the expressions:

$$\sum_{ss'} [\bar{u}(a)\Gamma_1 v(b)]\,[\bar{u}(a)\Gamma_2 v(b)]^* = \mathrm{Tr}[\Gamma_1(\not{p}_b - m_b)\bar{\Gamma}_2(\not{p}_a + m_a)]$$
$$\sum_{ss'} [\bar{v}(a)\Gamma_1 u(b)]\,[\bar{v}(a)\Gamma_2 u(b)]^* = \mathrm{Tr}[\Gamma_1(\not{p}_b + m_b)\bar{\Gamma}_2(\not{p}_a - m_a)]$$

[3] see the book "Introduction to Particle Physics", D. J. Griffiths, Pearson 2010, pp. 249 and 270.

THE DECAY $^{14}O \to {}^{14}N^* e^+ \nu_e$

- We shall take as a detailed example in the V−A theory the disintegration $^{14}O \to {}^{14}N^* e^+ \nu_e$, a β^+ emitter utilised in radiophysics [4], with lifetime τln2=71s, defect mass $\Delta m = 1.81\ \mathrm{MeV}/c^2$, and zero spin for both nuclei.
- We are going to assume that the underlying physical reaction is $p \to n e^+ \nu_e$ (the closest to $u \to d e^+ \nu_e$), although *kinematically only the nuclear reaction is exact*, the nuclei being quantum particles with well defined mass (see Problem 1).
- Assuming *energy and momentum conservation* in the above reaction, the partial width would be given by $d\Gamma = \frac{1}{2m} \int |\mathcal{M}|^2 dQ_3$, where $m = m_p$ is the proton mass, and dQ_3 the 3-dimensional Lorentz invariant phase-space volume, with $p_i = p_p$ and $p_f = p_n + p_e + p_\nu$:

$$dQ_3 = \frac{d^3\boldsymbol{p}_n}{(2\pi)^3 2E_n} \frac{d^3\boldsymbol{p}_e}{(2\pi)^3 2E_p} \frac{d^3\boldsymbol{p}_\nu}{(2\pi)^3 2E_\nu} (2\pi)^4 \delta^{(4)}(p_i - p_f) = \frac{d^3\boldsymbol{p}_e}{(2\pi)^3 2E_p} \frac{d^3\boldsymbol{p}_\nu}{(2\pi)^3 2E_\nu} \frac{d^3\boldsymbol{p}_n}{2E_n} 2\pi \delta^{(4)}(p_i - p_f)$$

- However, it is clear that *such conservation cannot happen*, since $m_p < m_n$ and the reaction only takes place thanks to the Fermi motion of the nucleons, the masses of p and n being ill-defined inside the nucleus (and even more so the u- and d-quark masses). Independently, the motion of p and n are deeply non relativistic.

[4] see the book F. Halzen and A. D. Martin, "Quarks and Leptons", John Wiley & Sons (1983).

THE KINEMATICS OF β-DECAY

- In order to obtain the exact kinematics, we should recall that the origin of the $\delta^{(4)}(p_i - p_f)$ factor is no other than the d^4x integration of $\mathcal{M}$ *including the participant plane waves*:

$$T_{if} = \frac{4G_F}{\sqrt{2}} \int \left[\bar{\psi}_n(x)\frac{1}{2}\gamma^\mu(1-\gamma^5)\psi_p(x)\right]\left[\bar{\psi}_\nu(x)\frac{1}{2}\gamma_\mu(1-\gamma^5)\psi_e(x)\right] d^4x$$

$$= \frac{G_F}{\sqrt{2}} \int \psi_n^+(x)\psi_p(x) \cdot \bar{u}_{p_\nu}\gamma^0(1-\gamma^5)v_{p_e} \cdot e^{-i(p_\nu+p_e)x} d^4x$$

- The $\psi_{p,n}$, involved in the short range $\mathrm{V-A}$ interaction, are contained within the nucleus, subject to the *relativistic* normalization condition $\int \psi_N^+\psi_N d^3\boldsymbol{x} = 2m_N$, where it only makes sense to talk about a *generic* wave function of a nucleon of mass m_N, given that m_p and m_n *are not well defined inside the nucleus*.
- As $\vec{v}/c \approx 0$ for the nucleon, $\vec{j} = \bar{\psi}_N\vec{\gamma}\,\psi_N = |N|^2\vec{v}/c \approx 0$, that implies $\psi_N = N(\uparrow, k\uparrow) \to N(\uparrow, 0)$, thus $\bar{\psi}_N\gamma^5\psi_N = 0$ and $\gamma^0\psi_N = \psi_N$. Hence all $\mu \neq 0$ indices become zero in the relativistic contraction, including the (e^+, ν_e) pair.
- The wavelength of the ν_e or e^+ $\lambda \sim h/(\Delta mc)$ is $\sim$ 100 times larger than the size of the nucleus ($\Delta m \sim 1\,\mathrm{MeV}/c^2$), thus we have $e^{i\boldsymbol{p}_\nu\boldsymbol{x}} \sim 1$ and $e^{i\boldsymbol{p}_e\boldsymbol{x}} \sim 1$ upon spatial integration over $d^3\boldsymbol{x}$ (note that only the nucleus inside contributes).
- The integration over t of the factor $e^{i(E_p-E_n-E_e-E_\nu)t}dt$ generates the exact energy conservation from Δm: a factor $2\pi\delta(\Delta m - E_e - E_\nu)$. Therefore:

$$\mathcal{M} = \frac{G_F}{\sqrt{2}}\left[\bar{u}_{p_\nu}\gamma^0(1-\gamma^5)v_{p_e}\right] 2m_N \cdot I_s$$

INSENSITIVITY TO NUCLEON MASS

- The isospin factor I_s is actually present in every β-decay process. In this case, $I_s = 2/\sqrt{2}$ (just compare $|pp\rangle$ in ^{14}O with $\frac{1}{\sqrt{2}}(|pn\rangle + |np\rangle)$ in $^{14}N^*$, with a factor 2 for the two identical protons that can decay).
- It is important to understand that the above result for Γ is insensitive to the nucleon mass values m_N, since on integration over the outgoing neutron phase-space:

$$d\Gamma = \frac{1}{2m_N}\int |\mathcal{M}|^2 dQ_3 = \frac{1}{2m_N}\int |\mathcal{M}|^2 \frac{d^3\boldsymbol{p}_N}{2E_N}(\dots)\ ,$$

the factor $1/(2m_N)^2$ $(E_N = m_N)$ exactly cancels the factor $(2m_N)^2$ in $|\mathcal{M}|^2$, because: $\int \frac{d^3\boldsymbol{p}_N}{2\sqrt{m_N^2 + \boldsymbol{p}_N{}^2}}\delta^{(3)}(\boldsymbol{p}_N + \boldsymbol{p}_{rec}) = 1/(2m_N)$ with $\boldsymbol{p}_{rec} = \boldsymbol{p}_p + \boldsymbol{p}_e + \boldsymbol{p}_\nu$ (an alternative calculation is proposed in Problem 1).

- The precise determination of the lifetime now requires to sum over the unobserved states of the positron (and, of course, of the neutrino) and the integration over the 6-dimensional phase-space of both momenta:

$$d\Gamma = \frac{I_s^2}{2}G_F^2 \sum_{\mathrm{ss}'} |\bar{u}(p_\nu)\gamma^0(1-\gamma^5)v(p_e)|^2 \frac{d^3\boldsymbol{p}_e}{(2\pi)^3 2E_e}\frac{d^3\boldsymbol{p}_\nu}{(2\pi)^3 2E_\nu} 2\pi\delta(\Delta m - E_e - E_\nu)$$

FERMI THEORY VERSUS V−A THEORY

- As we have seen, the chiral character of the V−A theory is still present in the factor $1-\gamma^5$, in spite of the non relativistic motion of the nucleons. There would be parity violation, should the positron polarization be measured. If we *give up measuring it*, the spin sum can be performed using Casimir's rule, and some properties of the traces of γ matrices, such as the following ($p_{1,2}$ being arbitrary 4-momenta):

$$\mathrm{Tr}\left[\gamma^\mu(1-\gamma^5)\not{p}_1\gamma^\nu(1-\gamma^5)\not{p}_2\right] = 8\left[p_1^\mu\cdot p_2^\nu + p_1^\nu\cdot p_2^\mu - (p_1\cdot p_2)g^{\mu\nu}\right] + 8i\epsilon^{\mu\alpha\nu\beta}p_{1\alpha}p_{2\beta}$$

- Indeed, on application of Casimir's rule with $\overline{\gamma^0(1-\gamma^5)} = \gamma^0(1-\gamma^5)$, the resulting trace is a particular case of the one above, with $\mu=\nu=0$:

$$\sum_{ss'}\left[\bar{u}_{p_\nu}\gamma^0(1-\gamma^5)v_{p_e}\right]\left[\bar{u}_{p_\nu}\gamma^0(1-\gamma^5)v_{p_e}\right]^* = \mathrm{Tr}\left[\gamma^0(1-\gamma^5)(\not{p}_\nu+m_\nu)\gamma^0(1-\gamma^5)(\not{p}_e-m_e)\right]$$
$$= 8(E_eE_\nu + \boldsymbol{p}_e\boldsymbol{p}_\nu) = 8E_eE_\nu(1+v_e\cos\theta)$$

  in the reference frame where the $^{14}\mathrm{O}$ is at rest.

- Note that $\mathrm{Tr}\left[\gamma^\mu\not{p}_1\gamma^\nu\not{p}_2\right] = 4\left[p_1^\mu\cdot p_2^\nu + p_1^\nu\cdot p_2^\mu - (p_1\cdot p_2)g^{\mu\nu}\right]$, which tells us that the Fermi theory would have produced an identical result to the one calculated above, since the factor $1/2$ in $G_F^2/2$ would not be present.

THE CURIE SPECTRUM

- Nowhere have m_ν and m_e been neglected, since the trace of a odd number of γ matrices is always zero, the matrices γ^5 and $1-\gamma^5$ being *themselves* an even number. Thus we have: $m_e \mathrm{Tr}[\gamma^0(1-\gamma^5)\not{p}_\nu(1+\gamma^5)\gamma^0] = 0$, and in addition $m_e m_\nu(1-\gamma^5)(1+\gamma^5) = 0$.
- In order to evaluate the (six-fold) integral in spherical coordinates, we take the Z-axis along the positron direction. Then $1 + v_e \cos\theta$ defines the angular distribution of the ν_e about the e^+, where the second term integrates to zero, irrespective of the actual velocity v_e. The highest probability is found when both particles come out *in the same direction*, as expected from angular momentum conservation, their helicities being *opposite* in the V$-$A theory, *both nuclei having spin zero*.
- The integrand in spherical coordinates thus takes the form (integrated azimuth):

$$d\Gamma = \frac{2G_F^2}{(2\pi)^5}(1 + v_e \cos\theta)\left[(2\pi d(\cos\theta) p_e^2 dp_e)(4\pi E_\nu^2 dE_\nu)\right]\delta(\Delta m - E_e - E_\nu)$$

- Leading to the positron spectrum, on integration over E_ν and $\cos\theta$ for the neutrino:

$$\frac{d\Gamma}{dp_e} = \frac{4G_F^2}{(2\pi)^3} p_e^2 (\Delta m - E_e)^2 \int_{-1}^{+1} d(\cos\theta)(1 + v_e\cos\theta) = \frac{G_F^2}{\pi^3} p_e^2 (\Delta m - E_e)^2$$

LIFETIME OF β DECAY

- The momentum distribution $d\Gamma/dp_e$ of β electrons is generically called Curie spectrum. Its maximum value $p_{\mathrm{e,max}}$ has been subject of investigation for many years, as a tentative means of detection of neutrino mass.
- We have seen in detail the reason why the Curie spectrum is identical in both the V−A theory and the Fermi theory: *despite the neutrino being ultrarelativistic, the nucleons are deeply non relativistic, and the electron polarization is not analysed*.
- If we just count the number emitted electrons per unit time interval in $(t, t+dt)$, we observe an exponential distribution: $N_0 e^{-t/\tau} = N_0 e^{-\Gamma t}$, as in every quantum decay process. The time constant τ indicates the emitter lifetime. The prediction for Γ is obtained upon integration of the previous expression for $d\Gamma/dp_e$:

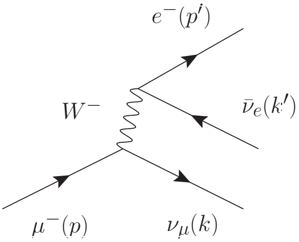
$$\Gamma = \frac{1}{\tau} = \frac{G_F^2 (\Delta m)^5}{30\pi^3}$$

  where for simplicity we have neglected m_e^2 as compared to $(\Delta m)^2$.
- The time analysis of many different β emitters provided a consistent and successful description of the data, with a unique Fermi constant G_F, independent of the $(\Delta m)^5$ value. However *precision* measurements require Coulomb corrections, due to the strong electric field seen by the positron (or electron) at the nucleus surface.
- Compare $G_F^\beta = 1.136(3) \times 10^{-5}\mathrm{GeV}^{-2}$ with $G_F = 1.1663788(7) \times 10^{-5}\mathrm{GeV}^{-2}$ from muon decay (next Lecture). A *physical* difference still remains, related to the quark mass matrices: V_{ud} or the Cabibbo angle, to be discussed in Lecture IX.

MUON DECAY

AMPLITUDE FOR $\mu^- \to e^- \bar{\nu}_e \nu_\mu$

$\boldsymbol{\mu^- \to e^- \bar{\nu}_e \nu_\mu}$



- Following the general structure $\frac{4G_F}{\sqrt{2}} J^\mu J^+_\mu$, we obtain the amplitude in the V−A theory:

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} \left[\bar{u}(k)\gamma^\mu(1-\gamma^5)u(p)\right] \left[\bar{u}(p')\gamma_\mu(1-\gamma^5)v(k')\right]$$

with $p = p' + k' + k$.

- Straightforwardly $\frac{1}{2}(1-\gamma^5)v(k')$ represents an outgoing $\bar{\nu}_e$ of *positive* helicity. Considering an incoming ν_e, with spinor $\frac{1}{2}(1-\gamma^5)u(k')$ in the Feynman diagram, would render the same result for $|\mathcal{M}|^2$.

- Every partial decay width follows the general quantum and relativistic expression $d\Gamma = \frac{1}{2m_A}\overline{|\mathcal{M}|^2} dQ$, m_A being the mass of the decaying particle (here $m_A = m_\mu$), the bar indicates *average* over all possible initial spin orientations and *sum* over all final spin states, and dQ is the Lorentz invariant phase-space volume element for the N final state particles ($A \to 1 + 2 + \cdots + N$).

THE 3-BODY PHASE-SPACE

- Labelling $\omega \equiv E_{\nu_\mu}$, $\omega' \equiv E_{\bar{\nu}_e}$, $E \equiv E_{\mu^-}$ and $E' \equiv E_{e^-}$, the generic structure of the 3-body phase-space is:

$$dQ = \frac{d^3\vec{\boldsymbol{p}'}}{(2\pi)^3 2E'} \left(\frac{d^3\vec{\boldsymbol{k}}}{(2\pi)^3 2\omega} \right) \frac{d^3\vec{\boldsymbol{k}'}}{(2\pi)^3 2\omega'} (2\pi)^4 \delta^{(4)}(p - p' - k - k')$$

$$= \frac{1}{(2\pi)^5} \frac{d^3\vec{\boldsymbol{p}'}}{2E'} \frac{d^3\vec{\boldsymbol{k}'}}{2\omega'} \delta\left((p - p' - k')^2\right)$$

- The last expression above is a consequence of the known integral over the relativistic mass shell: $\int \frac{d^3\vec{\boldsymbol{k}}}{2\omega} = \int d^4k\theta(\omega)\delta(k^2 - m_{\nu_\mu}^2)$, where $\theta(\omega)$ is the step function that selects only positive energies $\omega > 0$, and the fact that: $\int d^4k\delta^{(4)}(p - p' - k - k')\theta(\omega)\delta(k^2 - m_{\nu_\mu}^2) = \delta\left((p - p' - k')^2\right)$, with $m_{\nu_\mu}^2 = 0$.

- In a succint way, the integral over the mass shell is obtained as follows:

$\int d^4k\ \theta(\omega)\delta(k^2 - m^2) = \int d^3\boldsymbol{k} \int d\omega\ \theta(\omega)\delta(\omega^2 - m^2 - \boldsymbol{k}^2) =$
$\int d^3\boldsymbol{k} \int d\omega\ \theta(\omega) \frac{1}{|2\omega|} \left[\delta\left(\omega + \sqrt{m^2 + \boldsymbol{k}^2}\right) + \delta\left(\omega - \sqrt{m^2 + \boldsymbol{k}^2}\right)\right] = \int \frac{d^3\boldsymbol{k}}{2\omega} \cdot 1$

THE UNPOLARIZED AMPLITUDE

- If we *give up* doing the Garwin-Lederman experiment, and perform the *average* over the two spin orientations of the muon to calculate its lifetime, we may use Casimir's rule for each factor in $\mathcal{M} = \mathcal{M}_1\mathcal{M}_2$, by doing:

$$\overline{|\mathcal{M}|^2} = \frac{1}{2}\sum_{ss'}\mathcal{M}_1\mathcal{M}_2(\mathcal{M}_1\mathcal{M}_2)^* = \frac{1}{2}\left(\sum_s \mathcal{M}_1{\mathcal{M}_1}^*\right)\left(\sum_{s'} \mathcal{M}_2{\mathcal{M}_2}^*\right)$$

- Assuming $m_e = 0$ (and of course $m_{\bar{\nu}_e} = m_{\nu_\mu} = 0$) the trace takes the form:

$$\mathrm{Tr}\left[\gamma^\mu(1-\gamma^5)\not{k}\gamma^\nu(1-\gamma^5)(\not{p}+m_\mu)\right]\mathrm{Tr}\left[\gamma_\mu(1-\gamma^5)\not{p}'\gamma_\nu(1-\gamma^5)\not{k}'\right]$$

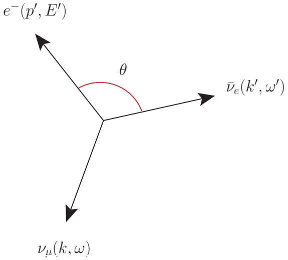
- At this point, it is extremely useful to know, for all V$-$A processes, the generic expression ($p_{1,2,3,4}$ being arbitrary 4-vectors):

$$\mathrm{Tr}\left[\gamma^\mu(1-\gamma^5)\not{p}_1\gamma^\nu(1-\gamma^5)\not{p}_2\right]\mathrm{Tr}\left[\gamma_\mu(1-\gamma^5)\not{p}_3\gamma_\nu(1-\gamma^5)\not{p}_4\right] = 256\,(p_1\cdot p_3)(p_2\cdot p_4)$$

- Leading to the partial result (the term in m_μ is an odd number of γ matrices):

$$\overline{|\mathcal{M}|^2} = \frac{G_F^2}{2}\;\frac{1}{2}\;256\;(kp')(pk') = G_F^2\;64\;(kp')\;(k'p)$$

THE 3-BODY KINEMATICS



- In the muon rest frame we have: $p = (m, 0, 0, 0)$ and $(p - k')^2 = (k + p')^2 = 2kp'$, since $p'^2 = m_e^2 = 0$ and $k^2 = m_{\nu_\mu}^2 = 0$ ($m \equiv m_\mu$).
- Given that $(p - k')^2 = p^2 - 2pk' = m^2 - 2m\omega'$ and $k'p = m\omega'$, we may write the V$-$A amplitude as: $2(kp')(k'p) = (p - k')^2(k'p) = (m^2 - 2m\omega')m\omega'$
- It becomes evident that the opening angle θ between the e^- and the $\bar{\nu}_e$ is univocally determined by the energies of both particles.

- Indeed, let us see how the above kinematic fact is precisely what is behind the Dirac δ-function: $\delta(k^2) = \delta(m_{\nu_\mu}^2)$ in the 3-body phase-space:

$$\delta\left((p - p' - k')^2\right) = \delta\left(p^2 - 2pp' - 2pk' + 2p'k'\right) = \delta\left(m^2 - 2mE' - 2m\omega' + 2E'\omega'(1 - \cos\theta)\right)$$

$$= \delta\left(A - 2E'\omega'\cos\theta\right) = \frac{1}{2E'\omega'}\delta\left(\cos\theta - \frac{A}{2E'\omega'}\right), \text{ with } A \equiv m^2 - 2m(E' + \omega') + 2E'\omega'$$

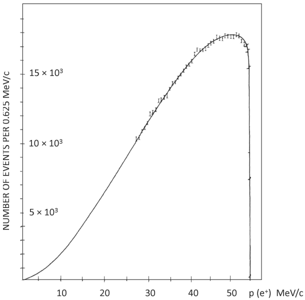
- As a result of which, the integrand of the partial width calculation can be written as:

$$d\Gamma = \frac{G_F^2}{2m\pi^5}\frac{d^3\boldsymbol{p'}}{2E'}\frac{d^3\boldsymbol{k'}}{2\omega'}m\omega'(m^2 - 2m\omega')\frac{1}{2E'\omega'}\delta\left(\cos\theta - \frac{A}{2E'\omega'}\right)$$

THE MICHEL SPECTRUM

- In order to perform the above (6-fold) integral over the e^- and $\bar{\nu}_e$ phase-space, we take the Z-axis of the spherical coordinates along the electron direction. Being unpolarized, both azimuthal distributions are flat, thus: $d^3\boldsymbol{p}'d^3\boldsymbol{k}' = 4\pi E'^2 dE' 2\pi\omega'^2 d\omega' d(\cos\theta)$. As the opening angle θ is fixed by the δ-function, it all boils down to a 2-fold integral over the energies:

$$d\Gamma = \frac{G_F^2}{2\pi^3} dE' d\omega' m\omega'(m - 2\omega') \quad \text{with limits:} \quad 0 \le E' \le \frac{m}{2} \quad \text{and} \quad \frac{m}{2} - E' \le \omega' \le \frac{m}{2}$$



Original measurements of the positron momentum, M. Bardon et al. Phys. Rev. Lett. 14, 449 (1965). The line corresponds to the derived formula for $d\Gamma/dE'$.

- Upon integration over ω' we obtain the Michel spectrum of energy (and momentum) of the emitted electron or positron:

$$\frac{d\Gamma}{dE'} = \frac{G_F^2}{12\pi^3} m_\mu^2 E'^2 \left(3 - \frac{4E'}{m_\mu}\right)$$

that historically constituted a stringent test on the V−A structure of the coupling. The excellent agreement with the data extends to recent high precision experiments.

- The mean muon lifetime τ_μ is readily obtained after integration over E':

$$\Gamma \equiv \frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

THE MUON LIFETIME

- When a given particle is allowed several decay channels, with different partial widths, $\Gamma_1, \Gamma_2, \ldots$, these add up to determine the total width: $\Gamma_t = \Gamma_1 + \Gamma_2 + \ldots$, with mean lifetime $\tau = 1/\Gamma_t$. The time elapsed before decay is still given by an exponential distribution with time constant τ, *no matter which channel is observed*.
- Given that the pion $\pi^-[\bar{u}d]$ is the lightest hadron, with higher mass than the muon, the decay $\mu^- \to e^- \, \bar{\nu}_e \, \nu_\mu$ is the *only channel* available. Indeed, a decay of the type: $\mu^- \to \bar{u} d \nu_\mu$ would result in the formation of at least one pion, since *free quarks are strongly unstable*. Furthermore, the electromagnetic decay $\mu^- \to e^- \gamma$ is *forbidden in the Standard Model*, as we shall see. Hence the calculated width amounts to the total width, the calculation of τ_μ being exact.
- In order to obtain the asymmetric distribution $1 - \frac{1}{3}\cos\theta$ for a polarized μ^- in the Garwin-Lederman experiment, it is necessary to use: $u^\uparrow \bar{u}^\uparrow = \frac{1}{2}(m + \not{p})(1 + \gamma^5 \not{s})$, with $s^\mu = (|\boldsymbol{p}|, 0, 0, E)/m$ (see Problem 2 and Exercises 19 and 20).
- The calculations shown in the V$-$A theory, where the 3 final state fermions have very small *squared* masses compared with the initial fermion mass ^{2}, constitute a perfect model to assess the decay of the τ lepton and that of the c, b and t quarks, allowing to estimate their mean lifetimes. Some examples: $\tau^- \to \nu_\tau d\bar{u}$, $c \to s\mu^+\nu_\mu$, $c \to su\bar{s}$, $b \to c\mu^-\bar{\nu}_\mu$, and $b \to cs\bar{c}$ (see Problems 3, 4, 5 and 6).

THE MUON AND THE G_F CONSTANT

- Recent measurements of τ_μ (D.M. Webber et al., PRL 106, 041803 (2011)) have allowed a very precise determination of one of the most important constants in physics, the Fermi constant $G_F = 1.1663788(7) \times 10^{-5}\mathrm{GeV}^{-2}$. Which requires to accumulate muons and study their Larmor precession frequency, following Garwin-Lederman's idea.
- If the $W^\pm$ boson mass is independently known (80.4 GeV/c^2), G_F fixes the true dimensionless coupling constant of the weak interaction g_w^2, as we have seen.
- Having a lifetime in its rest frame $\tau_\mu = 2196980.3(2.2) \times 10^{-12}s$, the muon lives in accelerators almost forever, in practical terms. It is the most penetrating charged particle known. Its selection at high energies requires the installation of a dense hadron absorber (typically Fe), with length in the meter scale.
- The highly penetrating power of the high energy muon ($E_\mu \gtrsim 5$ GeV) is a consequence of three unique features, leaving aside its long lifetime:
  - very low probability of bremsstrahlung (and pair creation), as compared to the electron (proportional to $1/m_\mu^2$).
  - low cross-section of deep inelastic (hadronic) scattering off nuclei ($\sigma_{\mathrm{inel}} \approx 0.1\, nb$), as compared to the pion.
  - cross-section of interaction with atomic electrons even lower than the above, although higher than for neutrinos (due to its electromagnetic character).
- The latter cross-sections are still lower for *neutrinos*, as will be seen (for their weak, not electromagnetic, interaction). However, the comparison is strongly energy dedendent.
- Almost all the great discoveries in particle physics have required the selection, or antiselection, of muons.

PION AND KAON DECAY

PIONS AND KAONS

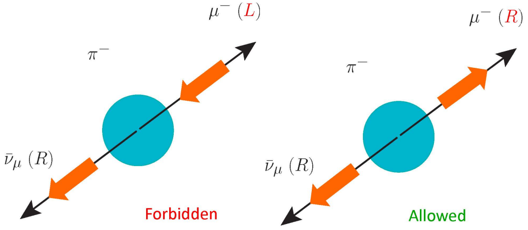
- Pions and kaons are the particles most frequently produced at accelerators. Their quark composition is $\pi^-[\bar{u}d]$, $K^-[\bar{u}s]$, and both have spin zero. Their electric charge is that of the electron, with masses $m_{\pi^\pm} = 139.6\ \mathrm{MeV}/c^2$ and $m_{K^\pm} = 493.7\ \mathrm{MeV}/c^2$. The neutral modes also exist, with masses: $m_{\pi^0} = 134.9\ \mathrm{MeV}/c^2$ and $m_{K_s^0} = 497.7\ \mathrm{MeV}/c^2$.

- At accelerators, the $\pi^\pm$ and $K^\pm$ are ultrarelativistic ($E_{\pi,K} \gtrsim 1\ \mathrm{GeV}$), and also penetrating particles (not as much as muons, but far more than electrons and photons), and they can traverse layers of matter of considerable thickness (detectors, absorbers). No industrial applications have been found for them, thus far.

- The relativistic $\pi^\pm$ and $K^\pm$ are penetrating for two fundamental reasons:
  - they fly long decay paths in the laboratory, due to Lorentz dilation ($\Delta x = c\tau_0\gamma$). Their lifetimes at rest are: $\tau_{\pi^-} = 26.0\,ns$ and $\tau_{K^-} = 12.4\,ns$ ($c\tau_0 = 7.8\,m$ and $3.7\,m$ respectively), that we shall study next.
  - because, like the muon, they do not undergo bremsstrahlung, also suppresed by their squared mass. This precludes the development of *electromagnetic showers*, that stops electrons and photons. They do develop hadronic showers due to their strong interaction with nuclei. The cross-sections are, however, much lower than (for instance) slow neutrons, thus generating *low nuclear activity* even though some radiological protection is required on the irradiated materials, for very intense beams.

THE HELICITY SUPPRESSION

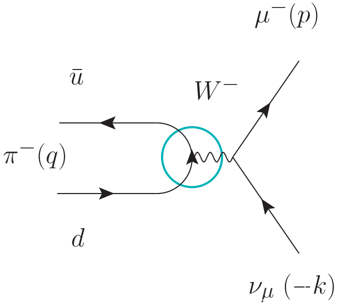
- Pions and kaons *decay into muons* by forming a charged current in the V−A theory: $\pi^- \to \mu^-\bar{\nu}_\mu$ and $K^- \to \mu^-\bar{\nu}_\mu$. Two facts call for specific attention, however:
    - why do they decay into muon and not into electron, with much larger phase-space?
    - and still decaying into muon, why the decay is 100 times faster than the muon decay itself, with smaller phase-space ($\propto m_\pi - m_\mu$ for the pion)?
- The answer to the first question is found precisely in the *chiral structure of the V−A coupling*. Indeed, *angular momentum conservation* tells us that the decay happens only through the opposite chirality (see right Figure below). As was seen, probabilities for both chiralities are: $P_{\text{opposite}} = (1-k)^2/[(1-k)^2 + (1+k)^2] = (1-\beta)/2$, with $\beta \equiv v_\mu/c$ and $P_{\text{equal}} = (1+k)^2/[(1-k)^2 + (1+k)^2] = (1+\beta)/2$. That explains why for the electron ($m_e = 0.5\,\text{MeV}/c^2$, $v_e/c \sim 1$) it is almost entirely suppressed.



- The second question requires a detailed analysis of the initial charged current in the pion.

THE f_π CONSTANT

- The $\bar{\nu}_\mu$ wavelength (or μ^-), $\lambda \simeq \frac{2h}{(m_\pi - m_\mu)c}$, is ~ 100 times larger than the π^- size ($\sim 1fm$). The pion is a bound state in QCD, and the leptonic charged current does not couple to the free quarks, but to the pion wave function, thus we cannot just assume a pure V−A coupling. This is in sharp contrast to the case of the $(\mu^-, \bar{\nu}_\mu)$ pair.



- The relativistic amplitude must contain a Lorentz vector, thus be proportional to q^μ (the only 4-vector available, with a spin zero pion): $(\ldots)^\mu = q^\mu f(q^2) \equiv q^\mu f_\pi$ [5]. The function $f(q^2)$ is the axial vector *form factor* of the pion (Fourier transform of the $\bar{u}d$ charge distribution), that must be evaluated at $q^2 = m_\pi^2$. So is the pion decay constant f_π defined.
- The constant f_π is known experimentally ($130.4\,\mathrm{MeV/c}$), and it can also be calculated in QCD, albeit with less precision.

- Given that $q = p + k$, the decay amplitude can be expressed as:

$$\mathcal{M} = \frac{G_F}{\sqrt{2}}\,(p^\mu + k^\mu)\, f_\pi \left[\bar{u}(p)\gamma_\mu(1-\gamma^5)v(k)\right] = \frac{G_F}{\sqrt{2}} f_\pi m_\mu \bar{u}(p)(1-\gamma^5)v(k)$$

[5] the pion being a *negative* parity state, this vector is proportional to the matrix element of the *axial* current between the π^- state and the vacuum: $iq^\mu f_\pi = \langle 0|\bar{u}\gamma^\mu\gamma^5 d|\pi^-\rangle$, or $\bar{u}d$ wave function overlap.

PION AND CHIRALITY

- The r.h.s. of the above equation is derived from the fact that the μ^- and the $\bar{\nu}_\mu$ are *free* outgoing particles, both satisfying the Dirac equation: $\bar{u}(p)(\not{p} - m_\mu) = 0$ and $\not{k}v(k) = 0$. The spinor $(1-\gamma^5)v(k)$ represents a *right-handed* $\bar{\nu}_\mu$ coupled to a μ^- that is *also right-handed* (as shown in the spin drawing of p. 35).
- The amplitude $\mathcal{M}$ shows analitically what was previously seen: *only the opposite helicity contributes*, the amplitude being *zero* in the limit $m_\mu \to 0$ (helicity suppression).
- The lifetime calculation now requires the spin sum, and the phase-space integration, according to: $d\Gamma = (1/2m_\pi)\overline{|\mathcal{M}|^2}dQ$. The former is achieved by application of Casimir's rule (together with the cyclic property of the traces):

$$\sum_{ss'} \mathcal{M}\mathcal{M}^* = \sum_{ss'} \frac{G_F^2}{2} f_\pi^2 m_\mu^2 \left[\bar{u}(p)(1-\gamma^5)v(k)\right] \left[\bar{u}(p)(1-\gamma^5)v(k)\right]^*$$

$$= \frac{G_F^2}{2} f_\pi^2 m_\mu^2 \,\mathrm{Tr}\left[(\not{p} + m_\mu)(1-\gamma^5)\not{k}(1+\gamma^5)\right]$$

- After suppression of the terms with an odd number of γ matrices, and using $\mathrm{Tr}[\not{a}\not{b}\gamma^5] = 0$, we have: $\mathrm{Tr}\left[\not{p}\not{k}(1+\gamma^5)(1+\gamma^5) + m_\mu\not{k}(1+\gamma^5)(1+\gamma^5)\right] = \mathrm{Tr}(2\not{p}\not{k}) = 8(p\cdot k)$. Finally:

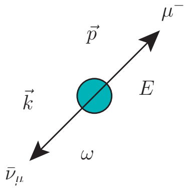
$$\overline{|\mathcal{M}|^2} = 4\cos^2\theta_c G_F^2 f_\pi^2 m_\mu^2 (p\cdot k)$$

- The amplitude includes a additional factor $V_{ud} = \cos\theta_c$, related to the quark mass mixing matrices, that we shall study later (see Lecture IX).

THE PION LIFETIME

- To perform the phase-space integration, we go to the pion rest frame. Taking the Z-axis along the μ^-, it is clear that both the momentum and the energy of the $\bar{\nu}_\mu$ are uniquely determined, the angular distribution being flat, in the absence of spin ($d^3\boldsymbol{k} = \omega^2 d\omega d\Omega$ with $\int d\Omega = 4\pi$):

$$p \cdot k = E\omega - \boldsymbol{k}\boldsymbol{p} = E\omega + \boldsymbol{k}^2 = \omega(E+\omega)$$



$$\Gamma = \left(\frac{G_F^2 f_\pi^2 m_\mu^2}{(2\pi)^2 2m_\pi}\right) \int \frac{d^3\boldsymbol{p}\, d^3\boldsymbol{k}}{E\omega} \delta(m_\pi - E - \omega)\delta^{(3)}(\boldsymbol{k}+\boldsymbol{p})\omega(\omega+E)$$

$$= (\,\ldots\,)\, 4\pi \int_0^\infty d\omega\ \omega^2(1+\frac{\omega}{E})\delta(m_\pi - E - \omega)$$

- The zero of the Dirac δ-function must be evaluated: $m_\pi - \sqrt{m_\mu + \omega_0^2} - \omega_0 = 0$ yielding $\omega_0 = (m_\pi^2 - m_\mu^2)/2m_\pi$, and we must take into account $\delta[f(\omega)] = \dfrac{1}{|f'(\omega_0)|}\delta(\omega - \omega_0)$, with $f'(\omega) = -(1+\frac{\omega}{E})$. Thus $\int_0^\infty d\omega\ \omega^2(1+\frac{\omega}{E})\dfrac{1}{(1+\frac{\omega}{E})}\delta(\omega-\omega_0) = \omega_0^2$, and finally:

$$\Gamma = \frac{1}{\tau_\pi} = \frac{G_F^2}{8\pi}\cos^2\theta_c f_\pi^2 m_\pi m_\mu^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2$$

PROPERTIES OF $\pi^- \to l^- \bar{\nu}_l$

- The experimental value of the $\pi^\pm$ mean lifetime in its rest frame is $\tau_\pi = 26.0\,ns$ ($c\tau_\pi = 7.8\,m$). Which is long enough to allow the magnetic measurement of its momentum at accelerators, while being in some cases too short to observe the pion-muon decay kink.
- The pion decay constant is thus known experimentally $f_\pi = 130.4\,\mathrm{MeV}/c$, which allows to understand τ_π quantitatively, thereby explaining why the pion actually decays faster than the muon.
- The electron decay $\pi^- \to e^- \bar{\nu}_e$ is much rarer, as we have seen, and the previous calculation in the V−A theory allows to make a clean prediction for the ratio of the respective probabilities, that is *independent of the value of* f_π:

  $$\frac{\Gamma(\pi^- \to e^- \bar{\nu}_e)}{\Gamma(\pi^- \to \mu^- \bar{\nu}_\mu)} = \left(\frac{m_e}{m_\mu}\right)^2 \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 = 1.283 \times 10^{-4}$$

  after using precision values of the 3 masses involved. The experimental value is $1.230(4) \times 10^{-4}$. Taking into account the orders of magnitude present in the suppression, a difference of only 3 % means a high precision test of the V−A coupling of the $W^\pm$ to both leptons.
- The above result *does not provide evidence*, however, of the V−A structure of the $W^\pm$ coupling to the *quarks*, owing to the long wavelength of the antineutrino, that does not couple to free quarks, but to the pion bound state as a whole.

THE KAON DECAY

- All previous results for the π^- apply also to the kaon $K^-[\bar{u}s]$: it is a bound state in QCD, where we have replaced the d quark by an s quark. The sizes of both mesons are not very different, as implied by $f_K \approx f_\pi$, in fact $f_K = 156.1(2)\mathrm{MeV}/c$. However, kaon mass is much higher: $m_{K^-} = 493.7\,\mathrm{MeV}/c^2$.
- The electron/muon suppression is even stronger, for the higher meson mass: 2.57×10^{-5}.
- Kaon lifetime $\tau_{K^-} = 12.4\,ns$ can also be understood in the V$-$A theory. The quark mass mixing matrices (that we shall study) now generate a factor $V_{us} = \sin\theta_c$ [6]. It is easily derived from the above that, for a generic lepton $l = \mu^-$ or e^-, we get the ratio:

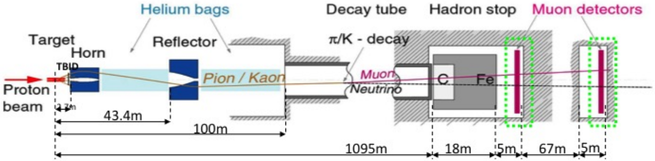
$$\frac{\Gamma(K^- \to l^-\bar{\nu}_l)}{\Gamma(\pi^- \to l^-\bar{\nu}_l)} = \tan^2\theta_c \left(\frac{f_K}{f_\pi}\right)^2 \left(\frac{m_\pi}{m_K}\right)^3 \left(\frac{m_K^2 - m_l^2}{m_\pi^2 - m_l^2}\right)^2$$

  Just because the chiral suppression was considered to be understood, the above formula played an important role in the determination of the Cabibbo angle $\theta_c = 13.1^\circ$, after the discovery of the strange particles in the 1960 decade.
- For the π^- there is no other lepton (apart from $l = \mu^-, e^-$), or hadron, with lower mass, thus the partial width $\Gamma(\pi^- \to \mu^-\bar{\nu}_\mu)$ is practically the total width. However the K^- mass allows the hadronic decays $K^- \to \pi^-\pi^0, \pi^-\pi^-\pi^+, \pi^-\pi^0\pi^0$, and the semileptonic decay $K^- \to \pi^0\mu^-\bar{\nu}_\mu$. All of them are weak decays and can be dealt with as charged currents in the V$-$A theory.

[6] note that we have nothing similar to the *lepton number conservation* for the quarks. Instead we have the Cabibbo angle. We shall come back to this topic in Lecture IX.

THE CREATION OF NEUTRINO BEAMS

- As in any other quantum decay process, we find an exponential time distribution, where the maximum probability density occurs at $t = 0$: the collision vertex where the $\pi^{\pm}$ and $K^{\pm}$ are produced.
- The fact that pions and kaons decay into muons, and almost never into electrons, is a determining factor for particle physics experiments: the noise over the *prompt* muons (those produced at the interaction vertex) will depend on energy, and on the amount of obsorber.
- The $\pi^{\pm}$ and $K^{\pm}$ decays are the basis for the construction of intense neutrino and antineutrino beams (ν_μ,$\bar{\nu}_\mu$) at any proton collider. Hadron focalization and absorption actually define the figure of merit of the neutrino beam, along with the intensity and degree of focalization of the primary proton beam.
- As derived from all the properties discussed above, muonic neutrino beams show a tiny contamination from electron neutrinos ν_e,$\bar{\nu}_e$ $(10^{-4} - 10^{-5})$. In addition, they allow to *selectively* create *neutrinos* ν_μ or *antineutrinos* $\bar{\nu}_\mu$, just by appropriate setting of the magnet polarities (π^+/π^-), as shown below:



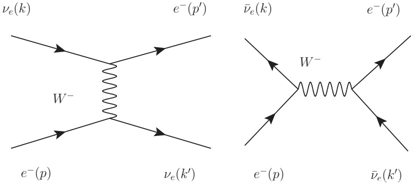
Setup of the CERN neutrino beams, directed towards the Gran Sasso (Italy) CERN-AC Note (2000-03)

Lecture V

NEUTRINO SCATTERING

ELASTIC AMPLITUDES $\nu_e e^-$ AND $\bar{\nu}_e e^-$

- Let us consider the elastic reactions $\nu_e e^- \to \nu_e e^-$ and $\bar{\nu}_e e^- \to \bar{\nu}_e e^-$, governed by charged V−A currents. The Feynman diagrams may be called $W^\pm$ *exchange* and $W^\pm$ *annihilation*, respectively:



- Both of them receive additional contributions from neutral currents, that we shall see in next lecture. The versions with *equal amplitude*: $\nu_\mu e^- \to \mu^- \nu_e$ and $\bar{\nu}_e e^- \to \bar{\nu}_\mu \mu^-$ instead, contain only charged currents. They both can be considered as the inverse muon decay.

- Their respective amplitudes are readily obtained from the diagrams:

$$\mathcal{M}_\nu = \frac{G_F}{\sqrt{2}} \left[\bar{u}(k')\gamma^\mu(1-\gamma^5)u(p)\right] \left[\bar{u}(p')\gamma_\mu(1-\gamma^5)u(k)\right]$$

$$\mathcal{M}_{\bar{\nu}} = \frac{G_F}{\sqrt{2}} \left[\bar{v}(k)\gamma^\mu(1-\gamma^5)u(p)\right] \left[\bar{u}(p')\gamma_\mu(1-\gamma^5)v(k')\right]$$

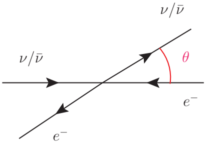
- It can be shown that the differential cross section for *any unpolarized two-body process* $ab \to cd$, follows the generic expression below, in the ultrarelativistic limit:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\overline{|\mathcal{M}|^2}$$

where the bar means *average* over all initial spin states, and *sum* over final spin configurations.

THE BACKWARD HELICITY SUPPRESSION

- On application of the Casimir rule and trace theorem already used for μ^- decay:

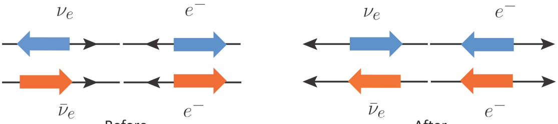


$$\frac{1}{2}\sum_{ss'}|\mathcal{M}_\nu|^2 = G_F^2 \cdot 64(kp)(k'p') = 16G_F^2 s^2$$

$$\frac{1}{2}\sum_{ss'}|\mathcal{M}_{\bar{\nu}}|^2 = G_F^2 \cdot 64(k'p)(kp') = 16G_F^2 u^2 = 4G_F^2 s^2(1+\cos\theta)^2$$

given that $s \equiv (k+p)^2 = 2(kp) = 2(k'p')$ and $u \equiv (k-p')^2 = -2(kp')$. The change of order of the 4-vectors in the annihilation channel is actually equivalent to replacing the Mandelstam variable s by u. Both are related by the angle θ in the center-of-mass frame (CM): $u = -\frac{s}{2}(1+\cos\theta)$, in the limit $m_e^2 = m_\nu^2 = 0$, with θ as indicated in the figure.

- It is very instructive to perform the calculation of s in the laboratory frame (LAB) where the beams have been prepared, and see that it would be zero in the limit $m_e \to 0$: $s = (k+p)^2 = (E_\nu + m_e)^2 - \boldsymbol{p}_\nu^2 = 2m_e E_\nu$. This is a *relativistic effect* that prevents energy dissipation on a target that is too light.
- The fact that $\mathcal{M}_\nu$ is *isotropic* in the CM frame while $\mathcal{M}_{\bar{\nu}}$ is not, reflects the V−A structure of the coupling, through the *helicity suppression* of the backward hemisphere ($\theta = \pi$), as shown below, from angular momentum conservation:



Before After

ELASTIC CROSS-SECTIONS $\nu_e e^-$ AND $\bar{\nu}_e e^-$

- The above suppression directly reflects itself in a factor of 3 suppression of the antineutrino cross-section, as compared to the neutrino, after using the two-body phase-space formula previously seen:

$$\frac{d\sigma}{d\Omega}(\nu_e e^- \to \nu_e e^-) = \frac{G_F^2 s}{4\pi^2} \implies \sigma(\nu_e e^-) = \frac{G_F^2 s}{\pi}$$

$$\frac{d\sigma}{d\Omega}(\bar{\nu}_e e^- \to \bar{\nu}_e e^-) = \frac{G_F^2 s}{16\pi^2}(1+\cos\theta)^2 \implies \sigma(\bar{\nu}_e e^-) = \frac{1}{3}\sigma(\nu_e e^-)$$

- It is useful to get an idea of the magnitude of the neutrino cross-sections in the V$-$A theory, i.e., in the Standard Model. Utilising the expression seen for s in the LAB frame, the numerical values obtained are expressed by the formula:

$$\sigma(\nu_e e^-) = \frac{2G_F^2 m_e}{\pi} E_\nu = E_\nu(\mathrm{GeV}) \times 10^{-2} fb$$

- Which summarises the following two main features:
  - *The cross-section increases linearly with the energy of the neutrino beams*, and for $E_\nu = 1\,\mathrm{GeV}$ it takes the value $10^{-2} fb$ ($1\ fb = 10^{-6} nb = 10^{-43} m^2$).
  - *They also increase linearly with the mass of the target fermion*. Being a kinematic effect, the above rule may also be applied for quarks, protons, neutrons, or nuclei. However, we must take into account that, either the mass may differ from its value in a vacuum (case of the quarks, that acquire a dressed mass), or the coupling will not be strictly V$-$A (the other cases). Still, cross-sections may be estimated using the above formula, for hadronic targets of given mass.

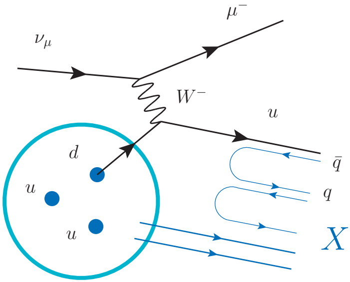
THE DEEP INELASTIC SCATTERING

Once the feasibility to build ν_μ and $\bar{\nu}_\mu$ beams with sufficient intensity was proven in the decade of 1970, at proton accelerators (J. Steinberger was pionier in this endeavour), these were used historically for 3 different scientific purposes [7]:

a) the study of the momentum distribution of quarks inside the proton and the neutron (partonic densities $q_i(x)$), with special attention to the antiquark density.

b) the study of weak interactions between the $W^\pm$ and the quarks, to find out whether they conform the V−A structure observed with the leptons.

c) the discovery, and first studies, of new forms of weak interaction, the neutral currents, as they were indeed brought to light by these experiments.



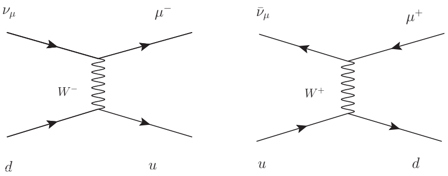
- Let us recall parton model's idea (Feynman, 1972), whose rationale is *independent of the nature of the interaction between the projectile and the quarks* (either electromagnetic or weak).

- The neutrino strikes the quark (or antiquark), in a short range reaction that produces its *transmutation* into another quark, and the emergence of a *muon*. The emitted quark cannot move away from the rest of the proton, due to the *color field* stretched between them, that dissipates the energy stored into an ensemble (X) of stable hadrons (pions, kaons, baryons, etc.). The hadronization process is practically *instantaneous* $(\sim 10^{-23} s)$, as compared with the time elapsed before the collision is fired by the weak interaction.

[7] neutrino beams are today essential to study neutrino masses, discovered in 2001. This topic is beyond the standard electroweak unification, and is not dealt with in this brief course.

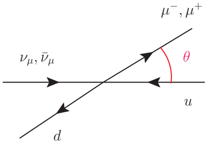
νq AND $\bar{\nu} q$ SCATTERING

- Neutrinos can only scatter off d-quarks in the proton, while antineutrinos only see the u-quarks, through the reactions $\nu_\mu d \to \mu^- u$ and $\bar{\nu}_\mu u \to \mu^+ d$, giving rise to the respective inclusive charged current processes: $\nu_\mu p \to \mu^- + X$ and $\bar{\nu}_\mu p \to \mu^+ + X$, corresponding to the Feynman diagrams:



- Note the processes: $\nu_\mu u \not\to \mu^+ d$ and $\bar{\nu}_\mu d \not\to \mu^- u$ *are forbidden*, since *no corresponding charged currents can be formed*, the neutrinos being blind to the u-quarks of the hadronic matter (antineutrinos to the d-quarks).

- By explicitating the corresponding spinors of the above amplitudes, for both processes, and making use of the Casimir rule, we find identical results to those previously obtained for $|\mathcal{M}_\nu|^2$ and $|\mathcal{M}_{\bar{\nu}}|^2$. Indeed, while the latter shows the characteristic suppression of the backward hemisphere, the former does not, as we have seen, owing to the V$-$A structure of the coupling. Thus the differential cross-sections in the CM frame (θ as indicated) are:



$$\frac{d\sigma}{d\Omega}(\nu_\mu d \to \mu^- u) = \frac{G_F^2 s}{4\pi^2}$$

$$\frac{d\sigma}{d\Omega}(\bar{\nu}_\mu u \to \mu^+ d) = \frac{G_F^2 s}{16\pi^2}(1 + \cos\theta)^2$$

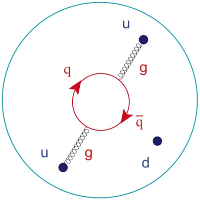
$\nu\bar{q}$ AND $\bar{\nu}\bar{q}$ SCATTERING

- Let us now hypothesize that neutrinos (and antineutrinos) *may also collide with antiquarks*, in case these are present inside the proton or neutron. It would not be necessary to repeat the calculations, because we are talking about the charge conjugate processes:

$$\frac{d\sigma}{d\Omega}(\bar{\nu}_\mu \bar{d} \to \mu^+ \bar{u}) = \frac{G_F^2 s}{4\pi^2} \quad \text{and} \quad \frac{d\sigma}{d\Omega}(\nu_\mu \bar{u} \to \mu^- \bar{d}) = \frac{G_F^2 s}{16\pi^2}(1+\cos\theta)^2$$

for which the $|\mathcal{M}|^2$ values are identical to the non conjugate ones. It is actually a CP transformation, in the V−A theory, where $\mathcal{M}$ can only differ by one phase (see Exercise 23). The total cross-section σ of the second process is $1/3$ of the first, as we have seen.

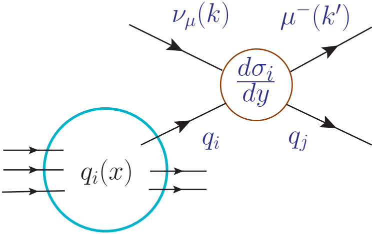
- The above hypothesis is plausible, given that the *small size of the proton* $(0.84 fm)$, implies fluctuations in the quark momenta: $\Delta p \sim \hbar/0.84 fm \sim 235\,\mathrm{MeV}/c$ that largely exceed the pair production threshold for u,d and s quarks $(2m_q c)$, according to the uncertainty principle.



- Hence the virtual pairs, produced by gluon exchange between quarks, have a sizeable probability of being struck by the neutrinos or antineutrinos, thus triggering an inelastic reaction. It is then understood why a detailed analysis of the *energy and scattering angle* of the outgoing muon $(\mu^\pm)$ has allowed to determine the *fraction of antimatter inside the proton*.

THE BJORKEN (x, y) VARIABLES

- In order to assess the exact relationship between the outgoing $\mu^\pm$ momentum in the LAB frame and the νq scattering angle in its CM frame, it is necessary to go back to Feynman's parton model, and know the precise meaning of the dimensionless variables: $(x, y) \in (0, 1)$ of Bjorken. These are uniquely determined by the 4-momenta of the incoming ν_μ (k), the outgoing $\mu^\pm$ (k') (their difference being $q = k - k'$), and that of the proton (p).



- After Feynman, $x \equiv -q^2/2(pq) = -q^2/(2M_p\nu)$ (with $\nu \equiv (pq)/M_p = E_{\nu_\mu} - E_{\mu^-}$), means the *fraction of the proton's longitudinal momentum carried by the struck quark, at the collision*.
- $y \equiv (pq)/(pk) = (E_{\nu_\mu} - E_{\mu^-})/E_{\nu_\mu}$ describes the νq *scattering angle* in their CM frame, according to $1 - y = (pk')/(pk) = (1 + \cos\theta)/2$, with $p = p_q/x$, as it can easily be checked.

- The key expression in Feynman's parton model is:

$$\frac{d\sigma}{dxdy}\left(\nu_\mu N \to \mu^- X\right) = \sum_i q_i(x)\left(\frac{d\sigma_i}{dy}\right)_{\hat{s}=xs}$$

- The differential cross-section of the emitted muon is determined by the *incoherent sum* of squared amplitudes over the different partons, 6 in this case: $i = u, d, s, \bar{u}, \bar{d}, \bar{s}$.The quantity $\hat{s}$ denotes the s value in the neutrino-quark reference frame. The gluons g are not seen by the neutrino, in a direct way.

THE ISOSCALAR TARGET

- In Feynman's formula, the following *4 fundamental processes* are defined, in the CM frame:

$$\frac{d\sigma}{dy}(\nu_\mu d \to \mu^- u) = \frac{G_F^2 xs}{\pi} \qquad \frac{d\sigma}{dy}(\nu_\mu \bar{u} \to \mu^- \bar{d}) = \frac{G_F^2 xs}{\pi}(1-y)^2$$

$$\frac{d\sigma}{dy}(\bar{\nu}_\mu u \to \mu^+ d) = \frac{G_F^2 xs}{\pi}(1-y)^2 \qquad \frac{d\sigma}{dy}(\bar{\nu}_\mu \bar{d} \to \mu^+ \bar{u}) = \frac{G_F^2 xs}{\pi}$$

- In most of the experiments, the matter interposed by the target had approximately equal number of protons and neutrons, or u-quarks and d-quarks (isoscalar target N, as marble (C,O) or Fe, with a small correction). A generic parton density was then defined, of quarks (Q) and antiquarks ($\bar{Q}$), taking into account the SU(3) symmetry:

$$Q(x) \equiv d^{\text{ proton}}(x) + d^{\text{ neutron}}(x) = d(x) + u(x)$$
$$\bar{Q}(x) \equiv \bar{u}^{\text{ proton}}(x) + \bar{u}^{\text{ neutron}}(x) = \bar{u}(x) + \bar{d}(x)$$

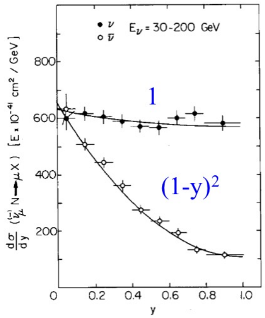
- The differential cross-sections (per nucleon) for ν_μ and $\bar{\nu}_\mu$ were determined from them:

$$\frac{d\sigma}{dxdy}(\nu_\mu N \to \mu^- X) = \frac{G_F^2 xs}{2\pi}\left[1\, Q(x) + (1-y)^2 \bar{Q}(x)\right]$$
$$\frac{d\sigma}{dxdy}(\bar{\nu}_\mu N \to \mu^+ X) = \frac{G_F^2 xs}{2\pi}\left[\bar{Q}(x) + (1-y)^2 Q(x)\right]$$

- The goal was to illuminate the target in a controlled way with ν_μ and $\bar{\nu}_\mu$, with precise knowledge of the beam energy $E_{\nu_\mu,\bar{\nu}_\mu}$, to identify the outgoing $\mu^\mp$, and to measure its momentum and scattering angle, as a means to determine (x, y) for each event. *It is not required* to identify or measure the hadron system X that recoils the nucleus.

THE V-A COUPLING OF $W^{\pm}$ TO QUARKS

- One of the most relevant results was obtained from detailed comparison of the y dependence of the ν_μ and $\bar{\nu}_\mu$ cross-sections, after x integration of the above expressions. Recall that the V$-$A structure of the $W^{\pm}$ coupling to quarks reflects itself in either a parabolic $(1-y)^2$ (backward suppresion), or constant 1 (isotropic scattering) behaviour. As $x \gtrsim 0.1$ is naturally selected, the antiquark contribution is small (but non zero, as we shall see).

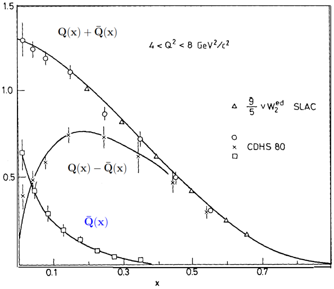


- It is clearly observed how the $\bar{\nu}_\mu$ scattering data on marble show the parabola $(1-y)^2$, while the ν_μ data show a constant distribution. The data show a residual contribution of antiquarks that slightly mixes both distributions.
- With $\lambda_{\nu,\bar{\nu}}$ (of De Broglie) being short enough, these data show the universal character of the chiral (V$-$A) structure of the $W^{\pm}$ coupling to quarks and leptons.

- Results published by the CHARM collaboration, M. Jonker et al., Phys. Lett. 109B (1982), 133., where a primary proton beam of 400 GeV was used, from the SPS at CERN.

THE ANTIMATTER FRACTION IN THE PROTON

- Another important result obtained with neutrino beams was the determination of the antimatter fraction inside the proton.
- While the electron (or muon) scattering experiments off protons are not able to distinguish quarks from antiquarks (for having equal electric charge squared), neutrino experiments are, as we have seen.



J. Steinberger, Nobel Lecture 1988.

- The fraction of antiquarks in the proton was determined to be: $\int_0^1 x\bar{Q}dx/\int_0^1 x(Q+\bar{Q})dx = (15 \pm 3)\,\%$, with $<-q^2>= 20\ (\mathrm{GeV/c})^2$ (J. G. H. De Groot et al., CDHS collaboration, Z. Phys. C1 (1979) 143.) [8]. Note the virtual $W^{\pm}$ wavelength $h/\sqrt{-q^2}$ amounts to approximately half the proton radius.
- Let us call $R = \sigma_{\bar{\nu}}/\sigma_{\nu}$ the ratio of total cross-sections for $\bar{\nu}_\mu$ and ν_μ, integrated in x and y. When antiquarks are not present ($\bar{Q}(x) = 0$), $R = 1/3$ is obtained. The fraction $r \equiv \int_0^1 x\bar{Q}(x)dx/\int_0^1 xQ(x)dx$ is thus related to the measured value of R through the expression: $r = (3R-1)/(3-R)$ (see Problem 9).

- The antiquark component in the proton was first observed in 1979 by neutrino experiments ($\nu_\mu/\bar{\nu}_\mu$), both at CERN with Fe target (CERN/Dortmund/Heidelberg/Saclay collaboration, CDHS), and at Fermilab with hydrogen target (Purdue/Argonne/Carnegie Mellon collaboration).

[8] this result includes the s-quark contribution.

Lecture VI

THE NEUTRAL CURRENTS

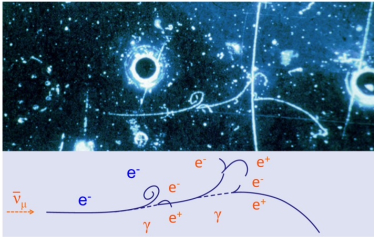
THE DISCOVERY OF NEUTRAL CURRENTS

- In 1973 neutrino interactions were discovered at the CERN Gargamelle bubble chamber, that could not be explained by the weak interactions known at that time. They belonged to the following two categories (neutrino energies $1 - 10$ GeV) :

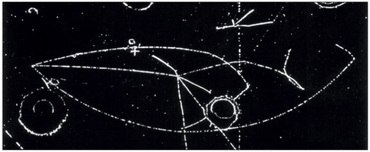
  a) $\bar{\nu}_\mu e^- \to \bar{\nu}_\mu e^-$ (later also $\nu_\mu e^- \to \nu_\mu e^-$).
  b) $\nu_\mu N \to \nu_\mu + X$ and $\bar{\nu}_\mu N \to \bar{\nu}_\mu + X$

  with N being a nucleus from the liquid Freon $(\mathrm{CF_3Br})$ filling the chamber, and X an ensemble of hadrons.

- Despite having been predicted by S. Weinberg in 1967 (as we shall see), important theoretical prejudices existed against the existence of weak neutral currents.



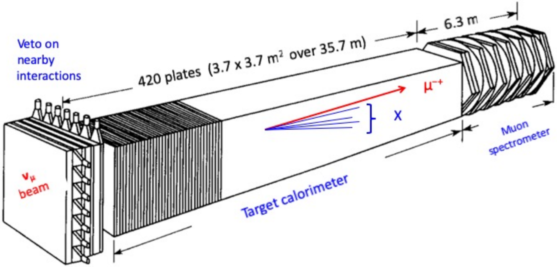
The first leptonic neutral current event observed in 1973. The antineutrino $\bar{\nu}_\mu$ incident from the left pushes an **electron** forward, creating a characteristic shower of bremsstrahlung photons and electron-positron pairs (*source*: CERN).



The neutrino interaction ν_μ from the left produces 3 charged particles, all of them identified as **hadrons**, since they interact with nuclei in the liquid filling the chamber. It is thus excluded that any of them can be a muon (*source*: CERN).

THE NEUTRINO EXPERIMENTS

- The CHARM detector is a example of the muon neutrino magnetic experiments that operated in the decade of 1980, associated with very high energy proton accelerators, both at CERN and at Fermilab.
- It is an *essential* feature of neutrino experiments to interpose a *large amount of matter in the target* (692 Tm in the case of CHARM) to compensate for the extremely low neutrino interaction cross-section. It also becomes necessary to veto neutron interactions.
- The goal is being sensitive to the charged currents (CC) $\nu_\mu N \to \mu^- + X$ and to the neutral currents (NC) $\nu_\mu N \to \nu_\mu + X$ simultaneously on the same apparatus, using *tagged* ν_μ and $\bar{\nu}_\mu$ beams (as earlier explained).
- Remarkably, the analysis of *neutral currents*, where no muon is present, requires measuring the energy deposited by the hadronic system E_X, in order to determine the y variable of Bjorken. The independent observation of charged current events allows to *calibrate* the detector response for E_X.



RELATIVE PROPORTION OF NEUTRAL CURRENTS

- The ratios between inclusive cross-sections of neutral currents (NC) and charged currents (CC) were determined, after a number of new experiments, to be:

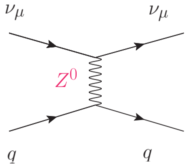
$$R_\nu \equiv \frac{\sigma^{NC}(\nu)}{\sigma^{CC}(\nu)} \equiv \frac{\sigma(\nu_\mu N \to \nu_\mu X)}{\sigma(\nu_\mu N \to \mu^- X)} = 0.31 \pm 0.01$$

$$R_{\bar\nu} \equiv \frac{\sigma^{NC}(\bar\nu)}{\sigma^{CC}(\bar\nu)} \equiv \frac{\sigma(\bar\nu_\mu N \to \bar\nu_\mu X)}{\sigma(\bar\nu_\mu N \to \mu^+ X)} = 0.38 \pm 0.02$$

- Being a new kind of weak interactions, the interest focused on the following questions: is the neutral current *chiral* (V−A) as the charged current? Is it coupled by the same Fermi constant G_F?
- The proportion of neutral currents was not small, as indicated above. At the beginning, these events were ignored, due to theoretical prejudices established at the time, related to the very small decay rate of $K_s^0 \to \mu^+\mu^-$. Actually, the latter had to do with the fact that neutral currents *do not change the quark flavor*, as we shall see later on (Lecture IX), as well as with the unexpected presence of *new quarks* (the charm c).
- Let us recall the definition of the Bjorken variable y, as calculated for a target at rest: $y \equiv (p.q)/(p.k) = (E_{\nu_\mu} - E'_{\nu_\mu})/E_{\nu_\mu} = E_X/E_{\nu_\mu}$, where $q = k - k'$ and E_X is the energy of the hadronic system. Noteworthy, it becomes necessary to measure E_X in the neutral currents, in addition to the neutrino beam energy E_{ν_μ}, needed for the kinematics.
- Note the significantly different values of R_ν and $R_{\bar\nu}$, that will be understood in the Standard Model (see Problem 10 of the course).

GENERIC AMPLITUDE FOR NEUTRAL CURRENTS

- The neutral currents (NC) are associated in Weinberg's theory with the exchange of a massive neutral boson, the Z^0, similar to the $W^\pm$ for the charged currents. The coupling constant is still proportional to the boson propagator for $q^2 \to 0$ ($q^2 \ll M_Z^2$):



$$\frac{g_n^2}{M_Z^2 - q^2} \to \frac{4G_F}{\sqrt{2}} \cdot f_n$$

- The following *empirical* amplitude is defined, for every generic NC process $\nu q \to \nu q$, as:

$$\mathcal{M}(\nu q \to \nu q) \equiv \frac{4G_F}{\sqrt{2}} 2\rho J_\mu^{NC} J^{NC,\mu} \quad \left(\text{recall for CC: } \mathcal{M} \equiv \frac{4G_F}{\sqrt{2}} J^\mu J_\mu^+\right)$$

with $\quad J_\mu^{NC}(\nu) \equiv \frac{1}{2}\bar{u}_\nu \gamma_\mu (C_V^\nu - C_A^\nu \gamma^5) u_\nu \quad$ and $\quad J_\mu^{NC}(q) \equiv \frac{1}{2}\bar{u}_q \gamma_\mu (C_V^q - C_A^q \gamma^5) u_q$

- In the above expressions, *the experiment must determine* the real *signed* constants $C_A^q, C_V^q \in \mathbb{R}$, to allow the coupling having V−A and V+A components, instead of being purely V−A (chiral), as for the charged currents.
- In addition, each quark type q (UP or DOWN) may have *different* couplings $C_{V,A}^q$, a situation which obviously does not happen with CC's. The neutrinos may be assumed to be purely left-handed, which would be natural, should they be massless, and produced from charged currents, with $C_V^\nu = C_A^\nu = 1/2$.
- The ρ constant amounts to a *redefinition* of C_V^q and C_A^q by a common factor, i.e. to an effective change of the G_F constant (factor f_n above). In view of a possible mass difference between the Z^0 and $W^\pm$ bosons, ρ must be precisely determined, and turns out to be a *key discriminant* for every lagrangian theory, together with the signed ratios C_A^q/C_V^q.

CROSS-SECTIONS ON ISOSCALAR TARGET

- In order to calculate $\nu q \to \nu q$ cross-sections with the above amplitude, it is important to realise that the characteristic suppression of the backward hemisphere, that we had already encountered in the V−A theory with the antineutrino, happens here again, *when the quark coupled to the current is right-handed* (and the neutrino left-handed), thus generating a term $(1-y)^2$.
- According to the standard definitions: $g_L^q \equiv \frac{1}{2}(C_V^q + C_A^q)$ and $g_R^q \equiv \frac{1}{2}(C_V^q - C_A^q)$, and the above-mentioned consideration, it is straightforward to derive, at the partonic level, the expression:

$$\frac{d\sigma}{dy}(\nu q \to \nu q) = \frac{G_F^2 x s}{\pi}\left[(g_L^q)^2 + (g_R^q)^2(1-y)^2\right]$$

- If we focus on the chirality [9], it is convenient to perform integration in x and define, for an isoscalar target N: $\mathbb{Q} \equiv \int_0^1 xQ(x)dx = \int_0^1 x[u(x)+d(x)]dx$ and $\bar{\mathbb{Q}} \equiv \int_0^1 x\bar{Q}(x)dx$, and the *average* couplings for u and d quarks: $g_L^2 \equiv (g_L^u)^2 + (g_L^d)^2$ and $g_R^2 \equiv (g_R^u)^2 + (g_R^d)^2$. The quantities $\mathbb{Q}$ and $\bar{\mathbb{Q}}$ may actually be determined with charged currents. Thus:

$$\frac{d\sigma^{\mathrm{NC}}}{dy}(\nu_\mu N \to \nu_\mu X) = \frac{G_F^2 s}{2\pi}\left\{g_L^2\left(\mathbb{Q} + (1-y)^2\bar{\mathbb{Q}}\right) + g_R^2\left(\bar{\mathbb{Q}} + (1-y)^2\mathbb{Q}\right)\right\}$$

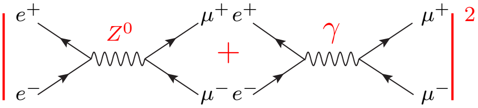
$$\frac{d\sigma^{\mathrm{NC}}}{dy}(\bar{\nu}_\mu N \to \bar{\nu}_\mu X) = \frac{G_F^2 s}{2\pi}\left\{g_L^2\left(\bar{\mathbb{Q}} + (1-y)^2\mathbb{Q}\right) + g_R^2\left(\mathbb{Q} + (1-y)^2\bar{\mathbb{Q}}\right)\right\}$$

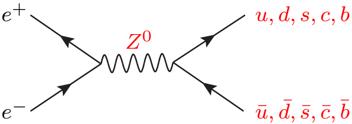
[9] and ignore a common factor ρ^2 in both terms

CHIRAL CONTENT OF THE NEUTRAL CURRENT

- The upshot of a whole generation of neutrino experiments, at the end of the 1980 decade, can be summarized by the results:
$$g_L^2 = 0.287 \pm 0.008 \qquad g_R^2 = 0.042 \pm 0.010$$
highlighting the fact that *the neutral current does not have the chiral character shown by the charged current*, unequivocally having a V+A component that the latter does not have. See Problem 8 for the numerical predictions in the Standard Model.

- The detailed study of neutral currents was key for the electroweak unification, and *it could not be performed by neutrino experiments alone*. On the one hand, a $p\bar{p}$ collider was built at CERN, to produce the Z^0 in $p\bar{p} \to Z^0 + X$. Simultaneously several generations of e^+e^- annihilation machines were built, that allowed to observe first the Z^0/γ interference and later to study the $C^q_{V,A}$, $C^l_{V,A}$, and $C^\nu_{V,A}$ couplings for all quarks, charged leptons l, and for the neutrinos:





For the Feynman diagram calculations, you may assess Problem 14 and Exercise 18, and Problem 13 and Exercise 21 for the neutrinos.

- Among the multiple lagrangian theories, based on the local gauge invariance principle, that tried to explain the precision data of the above projects, only one succeeded: the electroweak unification theory by Glashow-Weinberg-Salam or Standard Model, that we shall study next.

Lecture VII

THE $\mathrm{SU(2)_L \times U(1)_Y}$ LAGRANGIAN THEORY

KEY RELATIVISTIC LAGRANGIANS

- Fermionic field of **Dirac** coupled to a massless spin 1 field A_μ ($m_A = 0$), through the local gauge invariance principle, with covariant derivative $\mathcal{D}_\mu \equiv \partial_\mu + \frac{ie}{\hbar c} A_\mu$:

$$\mathcal{L}/(\hbar c) = i\,\bar{\psi}\gamma^\mu \partial_\mu \psi - \left(\frac{mc}{\hbar}\right)\bar{\psi}\psi - \left(\frac{e}{\hbar c}\right)\left(\bar{\psi}\gamma^\mu\psi\right) A_\mu = i\,\bar{\psi}\gamma^\mu \mathcal{D}_\mu \psi - \left(\frac{mc}{\hbar}\right)\bar{\psi}\psi$$

$$\text{Euler-Lagrange equation: } \left(\not{p}c - e\not{A} - mc^2\right)\psi = 0 \;\text{ with }\; \Box A_\mu = \partial_\mu(\partial_\nu A^\nu) \;\text{(free Maxwell)}$$

- Real spin zero scalar field, of **Klein-Gordon**:

$$\mathcal{L}/(\hbar c)^2 = \tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{2}\left(\frac{m_\phi c}{\hbar}\right)^2 \phi^2$$

$$\text{Euler-Lagrange equation: } \left(\Box + \left(\frac{m_\phi c}{\hbar}\right)^2\right)\phi = 0$$

- Real spin 1 vector field of **Proca**, $A_\mu = (\phi/c, \boldsymbol{A})$, with nonzero mass $m_A \neq 0$:

$$\mathcal{L}/(\hbar c)^2 = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\left(\frac{m_A c}{\hbar}\right)^2 A_\mu A^\mu \;\text{ con }\; F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\text{Euler-Lagrange equations: } \left(\Box + \left(\frac{m_A c}{\hbar}\right)^2\right) A_\mu = 0 \;\text{ and }\; \boldsymbol{\partial_\mu A^\mu = 0}$$

$$\text{because: } \left(\Box + \left(\frac{m_A c}{\hbar}\right)^2\right) A_\mu = \partial_\mu\left(\partial_\nu A^\nu\right) \implies \partial_\nu A^\nu = 0$$

- The integral $\int \mathcal{L} d^4x$ is a *relativistic invariant*, and it is measured in units of action $[J \cdot s]$, where the Lagrangian density $\mathcal{L}$ is energy per unit volume $\mathcal{L}\,[\mathrm{Jm}^{-3}]$.
- The scalar or vector fields may take the form $w_\pm = (w_1 \pm i w_2)/\sqrt{2}$, with *real* $w_{1,2}$ fields following the above $\mathcal{L}$, representing *opposite charge* particles, with (common) *doble* mass ^{2}.

GAUGE THEORY APPROACH

- Let us build a lagrangian theory that may represent both the *weak interaction* we have seen, and the electromagnetism (QED), using the principle of *local gauge invariance*. In the theory, the bosons involved in the CC's ($W^{\pm}$), and in the NC's (Z^0), will be real particles, similar to the photon, that can be emitted or absorbed by the fermions.
- The structure of the CC's suggests the fermionic fields being doublets: $\binom{u}{d}$ or $\binom{\nu_e}{e^-}$, used interchangeably. Their symmetry is the $\mathrm{SU}(2)$ group, that assigns to them, as a conserved charge, the weak isospin ($\pm 1/2$). The maximal P-violation observed forces to select only the left-handed *chiral* components, under the projector $\gamma_L = \frac{1}{2}(1-\gamma^5)$.
- The 3 generators of the $\mathrm{SU}(2)$ group (Pauli matrices) are associated with 3 *massless* spin 1 bosons (due to the local gauge symmetry), analogous to the photon: W^+, W^- and W^3.
- The $\mathrm{SU}(2)_L$ theory is assumed universal, with no distinction among the 3 generations of quarks and leptons. We shall take the first generation of the leptonic sector as a *reference example*, although any other of the *6 possible cases* are equally valid.
- These purely left-handed fields *cannot* describe *all* weak interactions of leptons and quarks, because these couple to the *neutral current* with a V+A component, as we have seen. In addition they interact with the *photon* in a *symmetric* way V−A / V+A. Thus *the theory must also contain their right-handed chiral fields*, under the projector $\gamma_R = \frac{1}{2}(1+\gamma^5)$.
- The right-handed chiral fields: $u_R\,, d_R\;, \nu_R$ and e_R are in the theory *different particles* from the left-handed fields, having different conserved charges (isospin zero singlets). It will be a partial success of the theory to *predict* that the ν_R is the only fermion having all of its conserved charges *zero*, thus being kept out of all Feynman diagrams.

CONTENT OF THE SU(2)$_L$ THEORY

- Let us recall the Pauli matrices:
$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2) \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \; \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
- The *3 fermions* (chiral fermionic fields) of the theory are:
$$\chi_L \equiv \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \quad (\nu_e)_R \quad (e^-)_R \quad \text{or} \quad \chi_L \equiv \begin{pmatrix} u \\ d \end{pmatrix}_L \quad (u)_R \quad (d)_R$$
- It can easily be checked, from what we saw in previous lectures, that the charged and neutral currents that couple to the bosons take the form:
$$J_\mu^+(x) = \bar{\nu}_L \gamma_\mu e_L \, e^{i(p_f - p_i)x} = \bar{\chi}_L \gamma_\mu \boldsymbol{\sigma_+} \chi_L \, e^{i(p_f - p_i)x}$$
$$J_\mu^-(x) = \bar{e}_L \gamma_\mu \nu_L \, e^{i(p_f - p_i)x} = \bar{\chi}_L \gamma_\mu \boldsymbol{\sigma_-} \chi_L \, e^{i(p_f - p_i)x}$$
$$J_\mu^3(x) = \left(\tfrac{1}{2} \bar{\nu}_L \gamma_\mu \nu_L - \tfrac{1}{2} \bar{e}_L \gamma_\mu e_L \right) e^{i(p_f - p_i)x} = \bar{\chi}_L \gamma_\mu \tfrac{1}{2} \boldsymbol{\sigma_3} \chi_L \, e^{i(p_f - p_i)x}$$
- Which can be joined together in a single weak current: $\vec{J}_\mu = \bar{\chi}_L \gamma_\mu \frac{1}{2} \vec{\sigma} \chi_L \, e^{i(p_f - p_i)x}$, that couples to the *3 gauge bosons* through the lagrangian:
$$\mathcal{L}_L = \mathcal{L}_0 + \mathcal{L}_G - g\vec{J}^\mu \cdot \vec{W}_\mu \quad \mathcal{L}_0 = \bar{\chi}_L \left(i\gamma^\mu \partial_\mu - m \right) \chi_L \, e^{i(p_f - p_i)x} \quad \mathcal{L}_G = -\tfrac{1}{4} \vec{W}_{\mu\nu} \vec{W}^{\mu\nu}$$
- Note that: $\frac{1}{2}\vec{\sigma}\vec{W}_\mu = \frac{1}{\sqrt{2}} \left(\boldsymbol{\sigma_+} W_\mu^+ + \boldsymbol{\sigma_-} W_\mu^- \right) + \frac{1}{2} \boldsymbol{\sigma_3} W_\mu^3$ with $W_\mu^\pm \equiv \frac{1}{\sqrt{2}} \left(W_\mu^1 \mp i W_\mu^2 \right)$
- The kinetic terms $\mathcal{L}_G$ of the gauge fields $\vec{W}_\mu$ show self-coupling, owing to the way in which the covariant derivative $\mathcal{D}_\mu$ acts on the fields: $\vec{W}_{\mu\nu} = \partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu + g\vec{W}_\mu \times \vec{W}_\nu$, the last term being determined specifically for the (non abelian) SU(2) group.

LIMITATIONS OF THE $\mathrm{SU(2)_L}$ THEORY

- Leaving aside the self-coupling, we see that the above theory is just a "copy" of QED, with $\mathcal{L}_{QED} = -eJ^\mu(x)A_\mu(x)$, where we have replaced the coupling constant $e = \sqrt{4\pi\alpha}$ by another dimensionless constant g, and Maxwell's electromagnetic field $A_\mu(x)$ by 3 similar massless fields: $W^1_\mu(x), W^2_\mu(x)$ and $W^3_\mu(x)$.
- At each space-time point x, the gauge rotations act on the left-handed fermion doublet, as defined by 3 arbitrary functions $\vec{\alpha}(x)$: $G(x) = \exp\left(\frac{i}{2}\vec{\sigma}\cdot\vec{\alpha}(x)\right)$, such that the invariance of $\mathcal{L}_L$ is ensured by the gauge transformation of the fields: $\vec{W}'_\mu = \vec{W}_\mu - \frac{1}{g}\partial_\mu\vec{\alpha} - \vec{\alpha}\times\vec{W}_\mu$, the last term ($\times$-product) being again determined by the specific structure of the $\mathrm{SU}(2)$ group.
- It is essential to realize that fermion masses must be strictly zero: $m = 0$, since a Dirac mass term in the lagrangian: $m\bar{e}e = m(\bar{e}_R e_L + \bar{e}_L e_R)$ *would not be invariant under the above gauge transformations*. Indeed, it is evident that the e_L field adquires a ν_{eL} component under $G(x)$, while e_R remains constant, for being an $\mathrm{SU(2)_L}$ *singlet*. Note that such situation *does not happen* in Quantum Electrodynamics (QED), lacking any kind of chirality, where the electron may indeed have mass.
- Owing to the very nature of the gauge principle, the bosons may not have mass either. A Proca term $+\frac{1}{2}M^2W^\mu W_\mu$ in $\mathcal{L}_L$ would not remain invariant under the $G(x)$ rotation.
- A critical question now comes up: is the weak interaction theory above compatible with quantum electrodynamics (QED), that couples the *same* fermions with the photon? Or more precisely: is the sum $\mathcal{L} = \mathcal{L}_L + \mathcal{L}_{QED}$ invariant under all rotations of the gauge group $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Q$?

WEAK HYPERCHARGE IN $\mathrm{SU(2)_L \times U(1)_Y}$

- The answer to the above question is NEGATIVE. Indeed, let us see that the electromagnetic current that couples to the photon: $\frac{1}{|e|}J_\mu^{em} = -\bar{e}\gamma_\mu e = -\bar{e}_R\gamma_\mu e_R - \bar{e}_L\gamma_\mu e_L$ (where $q = -|e|$ is the electric charge) *is not invariant under the rotation* $G(x)$ of $\mathrm{SU}(2)_L$, due to the term $\bar{e}_L\gamma_\mu e_L$, in which e_L acquires an uncompensated ν_{eL} component.
- The KEY IDEA that allows a way out of this problem is to couple, instead of J_μ^{em} [10], the WEAK HYPERCHARGE current, that is indeed invariant under $\mathrm{SU}(2)_L$:

$$J_\mu^Y \equiv 2\left(J_\mu^{em} - J_\mu^3\right) = -2\left(\bar{e}_R\gamma_\mu e_R + \bar{e}_L\gamma_\mu e_L\right) - \left(\bar{\nu}_L\gamma_\mu\nu_L - \bar{e}_L\gamma_\mu e_L\right) = -2\left(\bar{e}_R\gamma_\mu e_R\right) - 1\left(\bar{\chi}_L\gamma_\mu\chi_L\right)$$

By observing these coefficients in detail, the weak hypercharges acquired by the fermions of the theory can be read off: for the right-handed electron *singlet* $Y = -2$, and for the left-handed *doublet* $\chi_L(\nu_e, e^-)$ $Y = -1$, where $Y = 2(Q - T^3)$, with Q in units of $|e|$.

- Driven by this idea, and introducing a *new massless vector field* B_μ (that replaces the photon), with a new coupling constant g', we get a theory that is gauge invariant under local rotations of the group $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$:

$$\mathcal{L}_1 = -g\vec{J}^\mu \cdot \vec{W}_\mu - \frac{g'}{2}(J^Y)^\mu B_\mu$$

that is, a Yang-Mills theory, where the invariance of $\mathcal{L}_1$ under $G(x)$ rotations of the $\mathrm{SU}(2)_L$ group is manifest in the expression above for J_μ^Y, that involves the object: $(\bar{\chi}_L\gamma_\mu\chi_L)$. Invariance under $\mathrm{U}(1)_Y$ is also evident, for local phase changes $e^{\pm iY\alpha(x)}$ of the fermion fields, with equal hypercharge Y, compensate each other.

[10] from now on, we simply write J_μ^{em} instead of $\frac{1}{|e|}J_\mu^{em}$ (units of $|e|$).

GAUGE BOSONS AND PHYSICAL BOSONS

- Now the following obvious questions come up: where is Maxwell's electromagnetic field A_μ (the photon) in this theory? What is the value of the electron charge e as a function of g and g'? Where is the Z_μ field that transmits the *true* neutral current (having a V+A component, as we know), and what are the V−A and V+A couplings that the neutral current has for quarks and leptons, in this theory?
- To answer the above questions, S. Weinberg postulates in 1967 that the *physical* neutral bosons we see in the laboratory *are not* the gauge bosons present in $\mathcal{L}_1$, but two orthogonal linear combinations of them:

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} = \begin{pmatrix} cW^3_\mu - sB_\mu \\ sW^3_\mu + cB_\mu \end{pmatrix}$$

  where $c \equiv \cos\theta_W$ and $s \equiv \sin\theta_W$, with θ_W being the Weinberg angle. In addition, Weinberg provides the exact Hamiltonian of the interaction that brings the gauge states to the physical states (the Higgs mechanism), that we shall study in the next lecture.
- The equation below provides detailed account of the previous questions:

$$\mathcal{L}_{\text{neut}} = \left(J^{3,\mu} \;\; , \tfrac{1}{2}J^{Y,\mu}\right) \begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix} \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} = \left(J^{NC,\mu} \;\; , eJ^{em,\mu}\right) \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}$$

- It can easily be checked, by solving the implicit 2×2 linear system above (see Problem 12) that this equation shows a *unique solution* for the unknowns $J^{NC,\mu}$ and e, namely:

$$J^{NC,\mu} = \frac{g}{c}\left(J^{3,\mu} - s^2 J^{em,\mu}\right) \quad e = sg = cg' \quad e = \frac{gg'}{\sqrt{g^2 + g'^2}}$$

THE MEANING OF ELECTROWEAK UNIFICATION

- The first of these equations ($J^{NC,\mu}$) tells us *why* the neutral current is not purely V−A in Nature: it is *contaminated by the electromagnetic current*, as a consequence of the gauge symmetry $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$, with a V+A part *predictable* and proportional to $\sin^2\theta_W$.
- The second equation (e) tells us the gauge boson couplings in the Feynman diagrams: $g = e/\sin\theta_W$ ($\mathrm{SU}(2)$) and $g' = e/\cos\theta_W$ ($\mathrm{U}(1)$), as a function of the electron charge. We see that the interaction is not *intrinsically weak with respect to electromagnetism*, but rather the opposite. While waiting to comprehend the boson masses in a realistic unification theory (to be seen in next lecture), we observe that an experimental determination of θ_W, together with G_F and e, would allow us to predict (*without understanding*) the $W^\pm$ mass, *but not the Z^0 mass* (as the ρ parameter, defined on p. 57, remains unconstrained).
- The previous equations constitute the *core* of the electroweak unification theory, within the Standard Model. We see, however, that the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ symmetry is not able to interpret through its own means the Fermi constant G_F, since the latter assumes high masses of the intermediate bosons (dimension GeV^{-2}), that the theory *cannot interpret*, owing to the gauge principle itself, that requires them to have zero mass. Given that the gauge coupling constant between the $W^\pm$ boson and the fermions is $g/\sqrt{2}$ (p. 63), the exact relationship follows: $G_F = \sqrt{2}g^2/(8M_W^2)$ [11], from the G_F definition on p. 15.
- Let us recall that the electron charge is represented by the fine structure constant, that is dimensionless: $\alpha = e^2/(4\pi\epsilon_0\hbar c) \simeq 1/137$. Its small value justifies a *perturbative* analysis in QED, that is *therefore also justified in the electroweak theory* ($\sin\theta_W$ not being too small). In units $\hbar = c = \epsilon_0 = 1$ we have $e = \sqrt{4\pi\alpha}$.

[11] without adopting $\hbar = c = 1$, it becomes: $G_F = \sqrt{2}g^2(\hbar c)^3/[8(M_W c^2)^2]$ (units Jm^3).

THE CONSERVED CHARGES IN $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}$

- The $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ theory presents 4 conserved charges, according to Noether's theorem. Within a given flavor generation, we may use $T = |\vec{T}|$, T^3, Q, and the weak hypercharge $Y = 2(Q - T^3)$, to distinguish the different chiral states of the quarks and leptons in the theory, as detailed in the following tables (recall Q in units of $|e|$):

LEPTONS	T	T^3	Q	Y
$\nu_{e,L}$	$1/2$	$1/2$	0	-1
e^-_L	$1/2$	$-1/2$	-1	-1
$\nu_{e,R}$	0	0	0	0
e^-_R	0	0	-1	-2

$$\chi_L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \quad (\nu_e)_R \quad (e^-)_R$$

QUARKS	T	T^3	Q	Y
u_L	$1/2$	$1/2$	$2/3$	$1/3$
d_L	$1/2$	$-1/2$	$-1/3$	$1/3$
u_R	0	0	$2/3$	$4/3$
d_R	0	0	$-1/3$	$-2/3$

$$\chi_L = \begin{pmatrix} u \\ d \end{pmatrix}_L \quad (u)_R \quad (d)_R$$

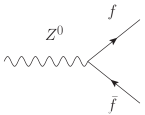
- The fermion/boson interaction part of the electroweak unification lagrangian is then:

$$\mathcal{L}_1 = -g\vec{J}^\mu \vec{W}_\mu - g' \tfrac{1}{2}(J^Y)^\mu B_\mu =$$

$$-\frac{g}{\sqrt{2}}(J^+)^\mu W^+_\mu - \frac{g}{\sqrt{2}}(J^-)^\mu W^-_\mu - e(J^{em})^\mu A_\mu - \frac{g}{\cos\theta_W}\left(J^3 - \sin^2\theta_W J^{em}\right)^\mu Z_\mu$$

THE Z^0 COUPLINGS

- The previous lagrangian and tables determine the Z^0 couplings in all Feynman diagrams, for each type of fermion f, by making explicit the C_V^f and C_A^f constants:



$$-\frac{g}{\cos\theta_W}\bar{\psi}_f\gamma^\mu\left[\tfrac{1}{2}(1-\gamma^5)T_f^3-\sin^2\theta_W Q\right]\psi_f\cdot Z_\mu$$
$$= \; -\frac{g}{\cos\theta_W}\bar{\psi}_f\gamma^\mu\tfrac{1}{2}\left(C_V^f-C_A^f\gamma^5\right)\psi_f\cdot Z_\mu$$

where the values $T^3=\pm 1/2$ should be used, corresponding to *left-handed* fermions.

- That are summarized in the table below, where $C_V^f=T_f^3-2\sin^2\theta_W Q_f$ and $C_A^f=T_f^3$:

f	Q_f	C_A^f	C_V^f
ν_e,ν_μ,ν_τ	0	1/2	1/2
e^-,μ^-,τ^-	-1	$-1/2$	$-1/2+2\sin^2\theta_W\simeq -0.03$
u,c,t	$2/3$	1/2	$1/2-\frac{4}{3}\sin^2\theta_W\simeq +0.19$
d,s,b	$-1/3$	$-1/2$	$-1/2+\frac{2}{3}\sin^2\theta_W\simeq -0.34$

from where the RH and LH couplings can be derived as: $g_{R,L}^f=(C_V^f\mp C_A^f)/2$.

- The value $\sin^2\theta_W\simeq 0.23$ has been used, as initially measured by neutrino experiments. Today we know that $\sin^2\theta_W=0.23120\pm 0.00015$.
- As we see, the predictive power of the $\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y$ theory is *huge*, and it explains why subsequent experiments performed at e^+e^- annihilation machines, in particular PETRA (DESY, Hamburg) and LEP (CERN), were able to exclude a great deal of alternative gauge theories.

Lecture VIII

THE ELECTROWEAK UNIFICATION THEORY

THE SPONTANEOUSLY BROKEN SYMMETRY

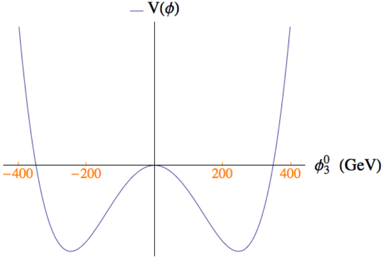
- Despite its success, the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ theory postulates, as we have seen, a LONG RANGE weak interaction, similar to electromagnetism, with massless bosons. However, the reality is quite different, with a weak interaction having SHORT RANGE, *in contrast* with electromagnetism, and characterized by a dimensionful Fermi constant G_F (GeV^{-2}). The gauge symmetry is therefore BROKEN in Nature. Note the interaction potential for a boson of mass m: $U(r) \propto -e^{-mr}/r$.
- The KEY IDEA on how to break a symmetry, while preserving it in $\mathcal{L}$, lies in attributing to a *vacuum* the responsibility of the broken symmetry, thus admitting the possibility that certain fields acquire there a nonzero value. This is *generically* known in field quantum theory as Spontaneous Symmetry Breaking (SSB).
- Y. Nambu (1960) is credited with this idea, actually staged 10 years earlier by L. D. Landau and V. Ginzburg in the context of giving the photon an effective mass inside a superconductor (1950). P. Higgs (1964), T. Kibble (1964), and others, clarified it in the relativistic and non abelian framework, and it was used in 1967 by S. Weinberg to solve the problem we are discussing here.
- Weinberg adds *4 real scalar fields* (spin zero) to the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ theory ($\mathcal{L}_1$), two of them electrically charged (± 1): ϕ_1^+ y ϕ_2^+, and two neutral ones: ϕ_3^0 y ϕ_4^0, in the form of a new isospin doublet ($T^3 = \pm 1/2$):

$$\phi(x) = \begin{pmatrix} \phi^+(x) \\ \phi^0(x) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1^+ + i\phi_2^+ \\ \phi_3^0 + i\phi_4^0 \end{pmatrix}$$

  it is immediately seen that, in this configuration, the 4 scalar particles have a weak hypercharge: $Y = +1$. The field ϕ_3^0 is forced to have a very high vacuum expectation value (VEV) v: $\phi_3^0(x) \equiv v + H(x)$.

THE HIGGS FIELD

- The above VEV is realized by postulating that the $\phi(x)$ field is *self-coupled*, with the potential energy: $V(\phi) = \mu^2 \left(\phi^+ \phi\right) + \lambda \left(\phi^+ \phi\right)^2$, where ϕ^+ now denotes the conjugate and transpose doublet, with two real constants: $\mu^2 < 0$ and $\lambda > 0$.



Clearly, a (degenerate) state of minimum energy is reached when the field (without particles) takes the constant value:

$$\phi_0 = \langle 0|\phi|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 + 0i \\ v + 0i \end{pmatrix}$$

with the relation $v^2 = \frac{-\mu^2}{2\lambda}$. We have plotted $V(\phi)$ over the ϕ_3^0 axis for the *observed* values of μ^2 and λ, that we shall know soon enough.

- The 4 new particles now interact with the $\vec{W}_\mu$ and B_μ bosons, by virtue of the local gauge invariance of $\mathcal{L}$, that *demands* the presence of the covariant derivative:

$$\mathcal{L}_2 = (\mathcal{D}_\mu \phi)^+ (\mathcal{D}^\mu \phi) + V(\phi) \quad \text{with} \quad \mathcal{D}_\mu = i\partial_\mu - \frac{g}{2}\vec{\sigma} \cdot \vec{W}_\mu - \frac{g'}{2} Y B_\mu$$

where $Y = +1$ is the hypercharge of ϕ. The gauge rotations $\mathrm{SU}(2)_L$ operate as:

$$\phi'(x) = e^{i\frac{\vec{\Lambda}(x)\cdot\vec{\sigma}}{2}} \phi(x) \qquad \vec{W}'_\mu(x) = \vec{W}_\mu(x) - \frac{1}{g}\partial_\mu \vec{\Lambda}(x) - \vec{\Lambda}(x) \times \vec{W}_\mu(x)$$

where $\vec{\Lambda}(x)$ are 3 arbitrary functions of space and time. The last term is dictated by the non abelian character of the $\mathrm{SU}(2)$ group, as already seen.

THE $W^{\pm}$ AND Z^0 MASSES

- The new lagrangian density is therefore $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$. Let us examine in detail how the interaction between the spin zero bosons and the gauge bosons takes place. The most immediate effect is obtained by replacing the $\phi(x)$ field by its vacuum expectation value $\phi_0 = (0, v/\sqrt{2})$, since the terms obtained will always be present in $\mathcal{L}$:

$$\mathcal{L}_2 = \left| \left(i\partial_\mu - \frac{g}{2}\vec{\sigma}\cdot\vec{W}_\mu - \frac{g'}{2} Y B_\mu \right)\phi \right|^2 + V(\phi)$$

$$\left| \cdots \right|^2 = \frac{1}{8}\left| \begin{pmatrix} gW^3_\mu + g'B_\mu & g\left(W^1_\mu - iW^2_\mu\right) \\ g\left(W^1_\mu + iW^2_\mu\right) & -gW^3_\mu + g'B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2$$

$$= \left(\tfrac{1}{2} v g\right)^2 W^+_\mu W^{-\mu} + \tfrac{1}{8} v^2 \begin{pmatrix} W^3_\mu & B_\mu \end{pmatrix} \begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix} \begin{pmatrix} W^{3,\mu} \\ B^\mu \end{pmatrix}$$

- The physical meaning of the first term is clear: the $W^{\pm}$ bosons have acquired a Proca mass, equal for both: $M_W = (1/2)vg$ [12] (not $(1/2)M_W^2 W^+_\mu W^{-\mu}$, but $M_W^2 W^+_\mu W^{-\mu}$ en $\mathcal{L}$). The second term shows *what the interaction mechanism is* between the W^3_μ and B_μ gauge bosons that gives rise to the physical bosons Z_μ and A_μ, as announced in the previous lecture. Indeed, on substitution of the expression defining the *Weinberg angle*, we obtain:

$$\tfrac{1}{8} v^2 \begin{pmatrix} Z_\mu & A_\mu \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix}$$

[12] without adopting $\hbar = c = 1$, the mass arises from the equation $(1/2)vg = M_W c^2$, resulting from $(1/2)vg/(\hbar c) = M_W c/\hbar$, after consistently expressing the gauge transformation.

THE FERMI CONSTANT

- We see that the purpose of the Weinberg angle is no other than *diagonalising* the above mass matrix. Its *eigenvalues* provide masses to the Z^0_μ (M_Z^2) and the photon A_μ (M_A^2):

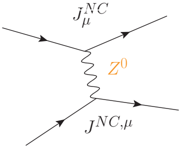
$$\frac{1}{8}v^2 \begin{pmatrix} Z_\mu & A_\mu \end{pmatrix} \begin{pmatrix} g^2 + g'^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix} = \frac{1}{2} M_Z^2 Z_\mu Z^\mu + \frac{1}{2} M_A^2 A_\mu A^\mu$$

with result: $M_Z = (v/2)\sqrt{g^2 + g'^2}$ and $M_A = 0$. $A_\mu = (g' W^3_\mu + g B_\mu)/\sqrt{g^2 + g'^2}$ is the eigenstate that has remained massless from Weinberg's choice of weak hypercharge, that we shall comment later on, $Z_\mu = (g W^3_\mu - g' B_\mu)/\sqrt{g^2 + g'^2}$ being the one with mass.

- Hence introducing a vacuum expectation value for one of the scalar fields ($\phi^0_3(x)$), of hypercharge $+1$, has given rise to a nonzero mass for the $W^\pm$ and Z^0 bosons, and the gauge symmetry has made both masses proportional to the *same* parameter v. Which leads to a specific prediction of this model for the ratio:

$$\frac{M_W}{M_Z} = \cos\theta_W$$

- Thus we have a *proper comprehension of the electroweak symmetry breaking*, that allows now to understand the Fermi constant G_F, and be able to calculate it precisely, just by going back to the definition of the generic coupling of the neutral current:



$$\mathcal{M}^{NC} \equiv \frac{4G_F}{\sqrt{2}} 2\rho J^{NC}_\mu J^{NC,\mu} = \left(\frac{g}{c} J^{NC}_\mu\right) \frac{1}{M_Z^2 - q^2} \left(\frac{g}{c} J^{NC,\mu}\right)$$

ρ PARAMETER, v VALUE AND HIGGS MASS M_H

- We identify $\rho \frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_Z^2 \cos^2\theta_W}$, and recall $\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}$ (Fermi constant definition on p. 67), to get the *important prediction for the ρ parameter*:

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2\theta_W} = 1$$

- Note this result goes BEYOND the predictions of the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ theory (that would imply $M_W = M_Z = 0$), and *could not have been achieved* without introducing the Higgs field and assigning it *specifically* to a doublet of hypercharge $+1$.
- Remarkably, the expectation value of the Higgs field v is uniquely determined by a single parameter, the Fermi constant $G_F = \frac{1}{\sqrt{2}} \frac{1}{v^2}$. Indeed: $\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} = \frac{g^2}{8(v^2 g^2/4)} = \frac{1}{2}\frac{1}{v^2}$, known with high precision from muon decay. A 3-digit value ready to memorise: $v = 246\,\mathrm{GeV}$. Note that its exact dimension is energy (GeV or J) [13].
- By replacing $\phi_3^0 = v + H(x)$ in the expression of $\mathcal{L}_2$, and taking into account $V(\phi) = \mu^2(v+H)^2 + \lambda(v+H)^4$, we see that a mass term is also produced for the Higgs boson: $\frac{1}{2}M_H^2 H^2 = (\mu^2 + 6\lambda v^2)H^2$, with $M_H^2 = -4\mu^2$ or $M_H = 2\sqrt{-\mu^2}$.
- The theory *does not tell us about the physical mechanism* behind the vacuum energy v of the $H(x)$ field, thus *no interpretation is provided for the λ and μ^2 parameters*, neither is the theory able to predict the *value* of the Higgs mass. The knowledge of v through G_F is not enough to fix these parameters. However, *an additional measurement of M_H does allow to fully determine, empirically, the potential energy $V(\phi)$*, fixing both λ and μ^2.

[13] the detailed relationship is $G_F = (1/\sqrt{2})(\hbar c)^3/v^2$ (Jm^3 units), without taking $\hbar = c = 1$.

THE MASSLESS SCALARS ARE DECOUPLED

- What happens to the other 3 spin zero bosons, charged and neutral: ϕ_1^+, ϕ_2^+ y ϕ_4^0? Detailed examination of expression $\mathcal{L}_2$ reveals their couplings in the Feynman diagrams of the theory. However, it is crucial to realize that, unlike the ϕ_3^0 field, their couplings may always be suppressed from the Feynman diagrams by an appropriate gauge rotation under the $\mathrm{SU}(2)_L$ group.
- Indeed, let us find the proper rotation to attain such goal, as defined by the three functions $\vec{\theta}(x)$ of space-time. We use the Pauli matrices to write: $\vec{\theta}\cdot\vec{\sigma} = \begin{pmatrix} \theta_3 & \theta_1 - i\theta_2 \\ \theta_1 + i\theta_2 & -\theta_3 \end{pmatrix}$ and the equation $\exp(i\alpha\vec{\theta}\cdot\vec{\sigma}) = \cos(\alpha|\vec{\theta}|)\mathbb{1} + i\sin(\alpha|\vec{\theta}|)\,(\vec{\theta}\cdot\vec{\sigma})/|\vec{\theta}|$. Then the most general infinitesimal rotation under $\mathrm{SU}(2)_L$ can be expressed as:

$$e^{i\frac{\vec{\theta}\cdot\vec{\sigma}}{\hbar c}}\,\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} = \frac{1}{\sqrt{2}}\,\big(v + H(x)\big)\begin{pmatrix} \frac{\theta_2(x)}{\hbar c} + i\,\frac{\theta_1(x)}{\hbar c} \\ 1 - i\,\frac{\theta_3(x)}{\hbar c} \end{pmatrix} \quad \text{for} \quad |\vec{\theta}| \to 0$$

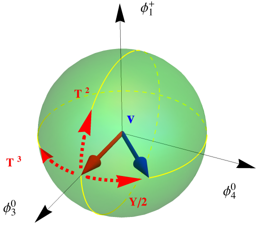
- It is evident that the expression of the r.h.s. runs through *all* possible configurations of the Higgs field $\phi(x)$ when $\vec{\theta}(x)$ runs over all possible sets of real functions $\theta_{1,2,3}(x)$. Note the rotation does not need to be infinitesimal, just for simplicity it is written in this way.
- Hence the *inverse* rotation: $e^{-i\frac{\vec{\theta}\cdot\vec{\sigma}}{\hbar c}}$ necessarily takes us to a gauge where *only the $H(x)$ boson couples* (called *unitary gauge*). For this reason the ϕ_3^0 particle is specifically called Higgs boson in the literature, and the other scalars *may be ignored* from the Feynman diagrams [14], in specific calculations. Naturally, the observables will never depend on the chosen gauge. For example, the mean lifetime of the t-quark in Problem 6, or the partial width into fermions of the Higgs boson, in Problem 17.

---

[14] the degrees of freedom represented by the 3 scalar fields actually become *extra* components of the *longitudinal* waves of the massive $W^{\pm}$ and Z^0 fields (see the formalism in the statement of Exercise 21).

THE PHOTON REMAINS MASSLESS

- Let us see more detail about the reason why the photon has remained masslesss ($M_A = 0$) in Weinberg's choice, and its relation with the conservation of the electric charge.



ϕ_2^+ *is removed from drawing (4D→3D)*

- None of the 4 generators of the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ group leaves invariant the VEV of the Higgs field:

$$\phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 + i0 \\ \mathbf{v} + i0 \end{pmatrix} .$$

The global $\mathrm{SO}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$ symmetry of $V(\phi)$ is broken by the above VEV to an $\mathrm{SO}(3) \simeq \mathrm{SU}(2)$ symmetry (*custodial symmetry*, in the literature). The hypercharge Y interaction ($g' \neq 0$) then introduces the photon in the theory, thereby inducing $\mathrm{M_W} \neq \mathrm{M_Z}$.

- The non invariance by any of the 3 rotations ($\vec{\sigma}$) of $\mathrm{SU}(2)_L$ is clear, according to the result of the previous page. For the $\mathrm{U}(1)_Y$ rotation (with $Y = +1$) it is also true:

$$e^{i\alpha(x)(+1)}\phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ e^{i\alpha(x)}\mathbf{v} \end{pmatrix}$$

- However, there is a rotation (linear combination of the above) which *does leave invariant that particular orientation of* ϕ_0, the one related to the conservation of the electric charge of quarks and leptons $Q = T^3 + \frac{1}{2}Y$ ($Q = \frac{1}{2}\sigma_3 + \frac{1}{2}\mathbb{1}$ on ϕ_0):

$$e^{iQ\theta}\begin{pmatrix} 0 \\ \mathbf{v} \end{pmatrix} = e^{i\frac{\sigma_3}{2}\theta} e^{i\frac{\mathbb{1}}{2}\theta}\begin{pmatrix} 0 \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}\begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}\begin{pmatrix} 0 \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 \\ e^{i\frac{1}{2}(\theta-\theta)}\mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{v} \end{pmatrix},$$

that explains why the linear combination $(g' W^3_\mu + g B_\mu)/\sqrt{g^2 + g'^2}$ remains massless. Choosing a VEV $\neq 0$ for the *charged* scalars would have implied electric charge violation.

HIGHLIGHTS OF ELECTROWEAK UNIFICATION

Some highlights that have been *decisive* in selecting the theory of electroweak unification we have studied, known as the Glashow-Weinberg-Salam theory (today part of the Standard Model of particle physics), are cited below:

- Discovery of the neutral currents by the Gargamelle experiment at CERN, 1974.
- Discovery of the $W^{\pm}$ boson and measurement of its mass $M_{W^{\pm}}$ at the upgraded SPS accelerator at CERN (running $p\bar{p}$ collisions with $\sqrt{s} = 540\,\mathrm{GeV}$) by the UA1 and UA2 experiments, 1983 ($M_{W^{\pm}} = 80.36 \pm 0.010\,\mathrm{GeV/c^2}$, current value (2024) by CMS).
- Discovery of the interference Z^0/γ in the charge asymmetry of $e^+e^- \to \mu^+\mu^-$ at PETRA (e^+e^- with $\sqrt{s} = 34.6\,\mathrm{GeV}$), DESY (Hamburg), by the experiments MARK J, JADE, and TASSO, and indirect determination of M_{Z^0}, 1983.
- Discovery of the Z^0 boson and measurement of M_{Z^0} at the upgraded SPS $p\bar{p}$ accelerator at CERN by the experiments UA1 and UA2, 1984.
- Precision measurement (10^{-4}) of M_{Z^0}, and of the Weinberg angle $\sin^2\theta_W$ from the asymmetries related to $C^f_{V,A}$, at LEP (e^+e^- with $\sqrt{s} = 91\,\mathrm{GeV}$) at CERN, by the experiments ALEPH, L3, DELPHI and OPAL (1991), and at SLAC by SLD (1998). Current values: $M_{Z^0} = 91.1876 \pm 0.0021\,\mathrm{GeV/c^2}$ and $\sin^2\theta_W = 0.23120 \pm 0.00015$ ($\overline{MS}$).
- Discovery of the Higgs boson and measurement of M_H at the LHC at CERN (pp with $\sqrt{s} = 7\,\mathrm{TeV}$), by the experiments ATLAS and CMS, 2012. Current value: $M_H = 125.09 \pm 0.21 \pm 0.11\,\mathrm{GeV/c^2}$.

Showing the *renormalizable* character of the electroweak theory ('t Hooft, 1971), intimately related to local gauge invariance, was a crucial step in the foregoing process. It means that calculations to a given order of the perturbative series of the Feynman diagrams, including loops, always provide a *finite* result.

THE FERMION MASSES

$\mathrm{SU(2)_L \times U(1)_Y}$ WITH 3 QUARK REPLICATES

- THE UP-type quarks (u, c and t, with $q = +\frac{2}{3}e$) and the DOWN-type quarks (d, s and b, with $q = -\frac{1}{3}e$) that couple in the electroweak theory, are actually *copies* within a flavor space of dimension 3 (the number of known replicates). Let us designate with capital letters their respective *massless* states, left-handed doublets and right-handed singlets:

$$\begin{pmatrix} U \\ D \end{pmatrix}_L, \; U_R, \; D_R \quad \text{with} \quad U_{L,R} \equiv \begin{pmatrix} \mathcal{U} \\ C \\ T \end{pmatrix}_{L,R} \quad D_{L,R} \equiv \begin{pmatrix} \mathcal{D} \\ S \\ B \end{pmatrix}_{L,R}$$

- Let us recall the 4 terms of the lagrangian density that describe their interactions with the photon (QED), with the $W^\pm$ bosons (charged currents) and with the Z^0 (neutral currents), using the above notation (remember $e = g s_w$):

$$\mathcal{L}_A^{(q)} = -e A_\mu J^\mu_{em} \quad \text{with} \quad J^\mu_{em} = \frac{2}{3}\left(\bar{U}_L \gamma^\mu U_L + \bar{U}_R \gamma^\mu U_R\right) - \frac{1}{3}\left(\bar{D}_L \gamma^\mu D_L + \bar{D}_R \gamma^\mu D_R\right)$$

$$\mathcal{L}_W^{(q)} = \frac{g}{\sqrt{2}}\left(W^+_\mu \bar{U}_L \gamma^\mu D_L + W^-_\mu \bar{D}_L \gamma^\mu U_L\right)$$

$$\mathcal{L}_Z^{(q)} = \frac{g}{c_w} Z_\mu \left(\frac{1}{2}\bar{U}_L \gamma^\mu U_L - \frac{1}{2}\bar{D}_L \gamma^\mu D_L - s_w^2 J^\mu_{em}\right)$$

where $c_w \equiv \cos\theta_W$ and $s_w \equiv \sin\theta_W$.

- Note that $\mathcal{L}_W^{(q)}$ contains a second term with the hermitic conjugate expression, that represents charged current transitions in the opposite direction (after time reversal).

THE YUKAWA COUPLING

- We have seen how the Higgs doublet $\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} 0 \\ (v + H(x))/\sqrt{2} \end{pmatrix}$ with a VEV of $v = 246\,\mathrm{GeV}$ and $Y = +1$ has generated masses for the $W^\pm$ and Z^0 bosons, thus breaking the electroweak symmetry. Let us now see how the above VEV of v also allows the *fermions to acquire mass*, beginning with the quarks.

- For that purpose, we need to know how the *charge conjugate* fields are expressed, representing the *antiparticles* of the Higgs bosons. We indicate here the solution, for a doublet of scalar fields ($\bar{\phi}^0 = \phi^{0*}$ for the individual fields) and go to the unitary gauge:

$$\phi_c = +i\sigma_2\phi^* = \begin{pmatrix} +\bar{\phi}^0 \\ -\phi^- \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} v + H(x) \\ 0 \end{pmatrix}$$

- The coupling of the Higgs fields ϕ to the fermion fields is *inevitable*, given that the following lagrangian density ($\mathcal{L}_{\text{Yukawa}}$) is *invariant under all local gauge rotations* of the group $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_Y$:

$$\mathcal{L}_{\text{Yukawa}} = -\left[G_d \left(\bar{U}_L \bar{D}_L \right) \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} D_R + G_u \left(\bar{U}_L \bar{D}_L \right) \begin{pmatrix} \bar{\phi}^0 \\ -\phi^- \end{pmatrix} U_R + \mathrm{H.C.} \right]$$

where $\mathrm{H.C.} = G_d^* \bar{D}_R \left(\phi^-,\ \bar{\phi}^0 \right) \begin{pmatrix} U_L \\ D_L \end{pmatrix} + G_u^* \bar{U}_R \left(\phi^0,\ -\phi^+ \right) \begin{pmatrix} U_L \\ D_L \end{pmatrix}$ denotes hermitic conjugate.

- Each term contains the product of 3 fields, so that the total weak hypercharge equals zero. For example: $Y = -1/3\,(\text{doublet}) + 1\,(\text{Higgs}) - 2/3\,(\text{singlet}) = 0$. This ensures its invariance under $\mathrm{U}(1)_Y$. Invariance under $\mathrm{SU}(2)_L$ is also clear, given that each term shows two opposite rotations, and one singlet that remains constant.

THE QUARK MASSES

- On first reading, we may think that the constants G_u and G_d are just numbers. But nothing prevents the presence of *mixed* terms where the L and R quarks belong to *different generations*, hence we are talking about 3×3 matrices. In addition, gauge invariance does not demand these numbers to be real, thus we are dealing with *complex* 3×3 *matrices*.
- Prior to investigating from $\mathcal{L}_{\text{Yukawa}}$ the coupling of the $H(x)$ boson to the quarks, let us simply replace the VEV of the fields in it (with $H(x) = 0$). It is clear that *mass terms* will show up for the DOWN- and UP-type quarks, in the form [15]:

$$\mathcal{L}_{\text{mass}} = -\bar{D}_L M_D D_R - \bar{U}_L M_U U_R + \text{H.C.} \quad \subset \quad \mathcal{L}_{\text{Yukawa}}$$

  which masses are, respectively: $M_D = vG_d$ and $M_U = vG_u$, arising from the *common* value $v = 246\,\text{GeV}$.

- Yet it is evident that the quark generations we observe in the laboratory are characterized by *measurable masses that are different for each generation*, hence the matrices M_D and M_U *must be diagonalizable*. In other words, *there exists an unknown Hamiltonian that creates these masses*, whose eigenstates are not the weak states, but linear combinations of them, that can be expressed by the *unitary matrices* $\mathcal{U}_{L,R}$ diagonalizing the Hamiltonian:

$$\mathcal{U}_L^{U+} M_U \, \mathcal{U}_R^U \equiv M_u = \text{diag}\,(m_u,\ m_c,\ m_t)$$

$$\mathcal{U}_L^{D+} M_D \, \mathcal{U}_R^D \equiv M_d = \text{diag}\,(m_d,\ m_s,\ m_b)$$

---

[15] recall that in Dirac theory $\mathcal{L}_M = -M_D \bar{D} D = -M_D(\bar{D}_L D_R + \bar{D}_R D_L)$ is the mass term.

MASS EIGENSTATES AND WEAK EIGENSTATES

- Notice that the (complex) quark mass matrices $M_{U,D}$ are not *hermitian*, thus they cannot be diagonalized by a *single* unitary matrix, instead *two of them*: $\mathcal{U}_L$ and $\mathcal{U}_R$ are required (biunitary diagonalization). The squares $M_{U,D}^+ M_{U,D}$ are indeed hermitian matrices.
- *Neither the theory predicts the quark masses, nor does it allow to establish relations among them*. The disparity observed through the 3 generations, from $m_u = 2\,\mathrm{MeV/c^2}$ to $m_t = 173000\,\mathrm{MeV/c^2}$, arising from a common origin in the VEV of the Higgs field ($M_{U,D} = v\, G_{u,d}$), raises an unsolved problem of *mass hierarchies*, since the values of $G_{u,d}$ alter v by many orders of magnitude.
- Let us designate by small letters the *mass eigenstates*, as it is customary, and use capital letters for the *flavor states* (or weak states). Then we can express the latter as a function of the former ones:

$$U_L = \mathcal{U}_L^U u_L \quad U_R = \mathcal{U}_R^U u_R \quad u_L \equiv \begin{pmatrix} \mathbb{u} \\ c \\ t \end{pmatrix}_L \quad u_R \equiv \begin{pmatrix} \mathbb{u} \\ c \\ t \end{pmatrix}_R$$

$$D_L = \mathcal{U}_L^D d_L \quad D_R = \mathcal{U}_R^D d_R \quad d_L \equiv \begin{pmatrix} \mathbb{d} \\ s \\ b \end{pmatrix}_L \quad d_R \equiv \begin{pmatrix} \mathbb{d} \\ s \\ b \end{pmatrix}_R$$

- The 4 matrices $\mathcal{U}_{L,R}^{U,D}$ are unitary ($\mathcal{U}^+ = \mathcal{U}^{-1}$), because they express in Quantum Mechanics a *change of basis* to transform the flavor states into the mass eigenstates of two *different* and *unknown* Hamiltonians: H^U and H^D ($H^U \neq H^D$). The massive quarks are therefore complex linear combinations among the different flavor generations.

THE KOBAYASHI-MASKAWA MATRIX

- Given that the quarks we observe in the charged currents, the neutral currents, and the electromagnetic currents, are *massive*, we must analyse the impact of the above rotations in all the corresponding observable processes.

- Beginning with the *charged currents*, we can express $\mathcal{L}_W^{(q)}$ right away as a function of the mass eigenstates::

$$\mathcal{L}_W^{(q)} = \frac{g}{\sqrt{2}} \left(W_\mu^+ \bar{u}_L \gamma^\mu V_{CKM} d_L + W_\mu^- \bar{d}_L \gamma^\mu V_{CKM}^+ u_L \right) \quad \text{with} \quad V_{CKM} \equiv \mathcal{U}_L^{U+} \mathcal{U}_L^D$$

  The matrix V_{CKM} was introduced by Kobayashi and Maskawa in 1973. It is a *unitary* and *complex* matrix of dimension $N \times N$ (the data suggest $N = 3$), that is usually written as:

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

- Should the UP-type (or DOWN-type) quarks be *massless*, the V_{CKM} matrix would not exist, since the mass eigenstates would be undefined, the rotation $\mathcal{U}_L^{U+}$ being arbitrary. Indeed, we could take $\mathcal{U}_L^{U+} = \mathcal{U}_L^{D+}$, thus making $V_{CKM} = \mathbb{1}$. This situation does indeed happen *almost* exactly in the *leptonic sector* with the neutrinos (UP-quark counterparts in the Yukawa couplings), which is why *we do not observe transitions between different generations* (so called conservation of the *leptonic number*).

- The V_{CKM} moduli contribute to all CC's processes, in particular to β-decay and pion decay: $|V_{ud}|^2$, to kaon decay: $|V_{us}|^2$, to heavy quark decays such as $b \to c\bar{u}s$: $|V_{cb}V_{us}^*|^2$ or $b \to u\mu^-\bar{\nu}_\mu$: $|V_{ub}|^2$, etc.

CABIBBO ANGLE AND WOLFENSTEIN'S PARAMETRIZATION

- A rotation matrix between the first two generations had already been postulated by Cabibbo and others in the 1960 decade, being later used by Glashow, Iliopoulos and Maiani to explain, by the presence of charm c, the strong suppression of $K_s^0\ [d\bar{s}] \to \mu^+\mu^-$.
- The first 2×2 box was written as: $\begin{pmatrix} V_{ud} & V_{uc} \\ V_{cd} & V_{cs} \end{pmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}$, with $\theta_c =$ Cabibbo angle. We have seen earlier how it was determined from $K^\pm \to l^-\bar{\nu}_l$ decays ($\theta_c = 13.1^\circ$).
- The pattern of V_{CKM} *moduli* reflects a LACK OF ALIGNMENT between the quark mass Hamiltonians $H^U \neq H^D$, and *no theory has been able to fundamentally interpret it to date*. After the discovery of the mean b-quark lifetime in 1984 at SLAC in the picosecond range, L. Wolfenstein clarified this pattern with an *approximate* parametrization of V_{CKM} (of Wolfenstein, in the literature), as simple powers of $\lambda \equiv \sin\theta_c$. In essence:

$$|V_{CKM}| \simeq \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}$$

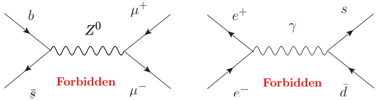
- What can we say about the *phases* of V_{CKM}? Individual phases of its $2N$ quarks are not measurable, except for a global phase. Being unitary, V_{CKM} has N^2 real parameters, of which $N^2 - (2N-1) = (N-1)^2$ are measurable. Should V_{CKM} be *real* (and unitary), it would have $\frac{1}{2}N(N-1)$ parameters. Therefore the total number of *measurable* and independent phases is: $\frac{1}{2}(N-1)(N-2)$ (the difference). For $N=3$, only one phase. Thus in the Standard Model V_{CKM} has 4 independent real parameters: *3 moduli and one phase.*
- As shown in 1973 by Kobayashi and Maskawa, this phase is the *only source* that is able to generate non conservation of the CP-symmetry in the electroweak theory we have seen (see Exercise 23 for more insight), with massless neutrinos. This is why this kind of processes are subject to intense investigation at different accelerators.

NEUTRAL CURRENTS DO NOT CHANGE FLAVOR

- Let us now see the implication of the quark mass matrix in the *neutral currents*, *including the electromagnetic current* (QED). We can write:

$$\begin{aligned}\mathcal{L}_Z^{(q)} &= \frac{g}{c_w} Z_\mu \left(\tfrac{1}{2} \bar{U}_L \gamma^\mu U_L - \tfrac{1}{2} \bar{D}_L \gamma^\mu D_L - s_w^2 J_{em}^\mu \right) \\ &= \frac{g}{c_w} Z_\mu \left(\tfrac{1}{2} \bar{u}_L \gamma^\mu u_L - \tfrac{1}{2} \bar{d}_L \gamma^\mu d_L - s_w^2 J_{em}^\mu \right)\end{aligned}$$

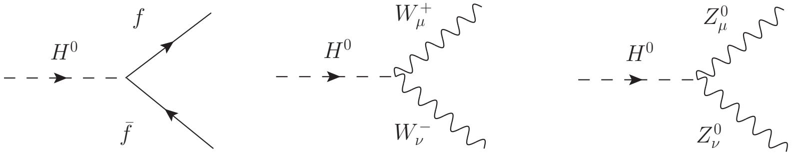
- The second equation is derived from the fact that the implicit flavor rotations in $\bar{U}_L$ and U_L are opposite. Indeed: $\bar{U}_L \gamma^\mu U_L = (\bar{u}_L \mathcal{U}_L^{U+}) \gamma^\mu (\mathcal{U}_L^U u_L) = (\mathcal{U}_L^{U+} \mathcal{U}_L^U) \bar{u}_L \gamma^\mu u_L$ with $\mathcal{U}_L^{U+} \mathcal{U}_L^U = \mathbb{1}$ from unitarity, and analogously for $\bar{D}_L \gamma^\mu D_L$. Note the cancellation also happens inside J_{em}^μ, hence in all photon couplings.
- The above derivation is *simple*, yet no less important its physical implication: *in the Standard Model, processes directly mediated by the Z^0, or by the photon, cannot change flavor*. When they do, they are called *flavor changing neutral currents (FCNC)*.
- For instance, the following processes are forbidden in the Standard Model, at lowest order: $b\bar{s} \to \mu^+\mu^-$, $e^+e^- \to s\bar{d}$, $Z^0 \to b\bar{s}$, the bremsstrahlung $b \to s\gamma$, etc. We show the diagramas for the first two (see Problem 15).



- Such processes may still occur in the Standard Model (SM) through quantum loops, with very small and *calculable* probabilities. They are being searched for in precision experiments, as a signal of new physics, beyond the SM. See Problems 7 and 16.

HIGGS COUPLING TO FERMIONS AND BOSONS

- By observing $vG_f(1+H/v)/\sqrt{2}$ in $\mathcal{L}_{\text{Yukawa}}$ (p. 81), we immediately see the Higgs boson coupling to fermion pairs in the Feynman diagrams. And doing the same in $\mathcal{L}_2$ (p. 73) we get from $(\mathcal{D}_\mu\phi)^+(\mathcal{D}^\mu\phi)$ a quadratic factor $\frac{1}{4}v^2g^2\left(1+H/v\right)^2W_\mu^+W^{-\mu}$, that leads to the coupling of H^0 to $W^\pm$ bosons, and analogously to Z^0 boson. See diagrams below:

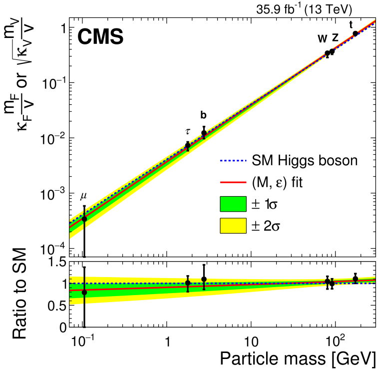


$-i\left(\frac{m_f}{v}\right)$ $\qquad$ $-2i\left(\frac{M_W^2}{v}\right)g_{\mu\nu} = -igM_W\,g_{\mu\nu}$ $\qquad$ $-i\left(\frac{M_Z^2}{v}\right)g_{\mu\nu} = \frac{-igM_Z}{2\cos\theta_W}\,g_{\mu\nu}$

- Hence the Higgs coupling becomes proportional to the mass in the fermion case, and to the mass squared in the boson case. The Higgs mechanism is, after all, responsible for the masses of all *elementary* bosons and fermions. This is a *very specific* feature of the electroweak symmetry breaking through the vacuum, in the Standard Model. Recall that $1/v = (\sqrt{2}G_F)^{1/2}$.
- Higgs boson decay into W^+W^- and Z^0Z^0 pairs *is* possible, despite the fact that one of the bosons lies below its mass shell. The $W^\pm$ and Z^0 bosons typically decay into leptons $(l_1, \bar{l}_2)$, and the calculation of the respective H^0 partial widths requires evaluating the boson propagator at $q^2(p_{l_1}, p_{\bar{l}_2})$, for example in the 3-body process $H^0 \to Wl_1\bar{l}_2$.
- We propose in Problem 17 the calculation of the partial width into fermions $\Gamma(H^0 \to f\bar{f})$.

EVIDENCE OF THE HIGGS COUPLINGS

- Proportionality to the mass and mass^2 of the Higgs couplings, as a specific feature of the electroweak symmetry breaking through the vacuum, in the Standard Model, has been made clear in 2018 to a remarkable level of precision by the ATLAS and CMS experiments (ATLAS-CONF-2018-031, CERN-EP-2018-263, arXiv:1809.10733v1), thereby confirming the predictions within a large range of variation of the fermion and boson masses.



- The t-quark coupling has been determined indirectly, by isolating a sample of events showing $t\bar{t}H^0$ associated production, accounting for the production $\sigma(pp \to H^0 X)$ and decay $H^0 \to \gamma\gamma$ rates, to which this coupling is very sensitive.
- Factors $k_{F,V}$ were determined for the H^0 couplings to fermions: $k_F(m_f/v)$ and bosons: $2k_V(m_W^2/v)$ $(W^\pm)$ or $k_V(m_Z^2/v)$ (Z^0), with $k_F \neq 1$ and $k_V \neq 1$ signaling any deviation from the Standard Model.
- The fitted values of k_F and $\sqrt{k_V}$ are shown, multiplied by factors $m_{F,V}/v$ (to illustrate the comparison), along with their uncertainty, with the value $v = 246\,\mathrm{GeV}$.

- These results reveal that the particle found in pp collisions with $\sqrt{s} = 13\,\mathrm{TeV}$ meets the properties of the Higgs boson studied here, with remarkable precision. Like for the Z^0 case (albeit less accurately) the data have not shown to date any evidence of internal structure.

PROBLEMS AND EXERCISES

of the course

Problem 1 ($\star$)

Phase-space of β-decay

Show that in the decay $N_1 \to N_2\, e^- \bar{\nu}_e$, with $N_{1,2}$ being *spinless* nuclei with masses $m_{1,2}$, with spin summed amplitude in the V−A theory $|\overline{\mathcal{M}}|^2$, the partial width $d\Gamma = \frac{1}{2m_1}|\overline{\mathcal{M}}|^2 dQ$ takes the exact form:

$$d\Gamma = \frac{G_F^2}{2} I_s^2 \sum_{\text{spins}} \left|\bar{u}(p)\gamma^0(1-\gamma^5)v(k)\right|^2 \frac{d^3\boldsymbol{p}}{(2\pi)^3 2E} \frac{d^3\boldsymbol{k}}{(2\pi)^3 2\omega} \delta(\Delta m - E - \omega) \cdot 2\pi$$

where $p = (E, \boldsymbol{p})$ and $k = (\omega, \boldsymbol{k})$ are the 4-momenta of the electron e^- and antineutrino $\bar{\nu}_e$, I_s a global isospin factor of the involved nuclei, dQ the invariant 3-body phase-space volume element, and $\Delta m = m_1 - m_2$ the mass defect.

The above expression should be proven differently from the way it was pursued in the course. Instead of assuming that the proton and the neutron are responsible for the decay, with $V - A$ coupling, simply assume that the spinless nuclei have a *scalar* relativistic coupling, proportional to the 4-vector $(p_1 + p_2)^\mu$, to be contracted with the leptonic charged current. When performing the integral over the 3-body phase-space, use the mass-shell integral of p. 27, and make explicit the function $\delta(p_2^2 - m_2^2)$, before taking the limit $m_1, m_2 \to \infty$. In this way, the exact kinematics is obtained, obviating the Fermi motion consideration. Furthermore, it is made clear that there is no need to involve the proton and the neutron (let alone the quarks) in β decay, in order to understand precisely the Curie spectrum. Of course, we have been forced to assume that both nuclei are spinless (which is certainly *not* the most general case).

Problem 2 ($\star\star\star$)

Garwin-Lederman's asymmetry

Show that in *polarized* muon decay $\mu^- \to e^- \nu_\mu \bar{\nu}_e$, the angular distribution of the electron takes the form dΓ/d(cosθ) $= A(1 - \frac{1}{3}$cos$\theta)$, where θ is the angle between the electron and the muon spin direction. It is suggested to follow the steps of the integration over the 3-body phase-space, as in the derivation of the unpolarized case made in the course, and then use the following angular integral, for $\alpha, \beta \in \mathbb{R}$, $|\alpha| \leq |\beta|$, with $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ being unitary vectors:

$$\int \boldsymbol{n}(\Omega) \cdot \boldsymbol{n}_1 \delta\Big(\alpha - \beta \boldsymbol{n}(\Omega) \cdot \boldsymbol{n}_2\Big) d\Omega = \frac{2\pi\alpha}{\beta^2} \boldsymbol{n}_1 \cdot \boldsymbol{n}_2$$

The factor $-1/3$ of the angular distribution is essential to understand the results of the historical parity violation experiment of Garwin-Lederman in 1957. The result of Exercises 19 and 20, relative to polarized fermions, is actually needed, and should be known, even if these are not assessed. In order to perform the sum over ν_μ spin in a given factor, just use the lemma: $\sum_s \bar{u}(k) \Gamma u(k) = \mathrm{Tr}\left[(\not{k} + m)\Gamma\right]$, where Γ is any product of gamma matrices, independent of s.

Problem 3 ($\star\star$)

τ-lepton mean lifetime

Use the exact calculation of the muon lifetime in the V−A theory to perform an approximate prediction of the mean lifetime of the τ-lepton (τ_τ) in ps, as function of its mass. Assess two different scenarios:

a) the τ may only decay into leptonic modes (electron or muon).

b) it may also decay into quarks, and each type of them should be assigned a partial width Γ_q, from a simple model that takes into account the color factor of 3, and the Cabibbo angle. Take into consideration the s-quark, indicating which mesons would be present in the final state in this case. May the τ decay into charm?

Compare the predictions with the tabulated results of the PDG. Do it first for the τ lifetime, and then for the leptonic and hadronic partial decay widths separately. Justify the hypothesis of zero mass your are doing for all final state fermions. Express the energy conservation in detail, for a given *mesonic* final state of your choice, and calculate the neutrino wavelength in the τ rest-frame, for that particular choice. Does the neutrino see the quarks, or the mesons? Is the $\mathrm{V}-\mathrm{A}$ coupling to the mesons garanteed? Comment freely on the precision attained by your estimate of τ-lifetime.

Problem 4 ($\star\star$)

Charm quark mean lifetime

a) The c-quark decays by virtue of its chiral coupling to charged currents. Apply what has been learned from $\mu^- \to e^- \nu_\mu \bar{\nu}_e$ to understand its mean lifetime in the range of the ps, as function of its mass. For that purpose write its 8 main decay modes into quarks and leptons and assign each of them a partial width, taking into account the Cabibbo angle. Make a drastic simplification of the color treatement by only assuming a factor 3 in those modes involving 2 hadronic charged currents. Verify that in all cases the squared mass of the final state fermions may be neglected.

b) In accordance with the previous results, ignoring all details of the hadronization process, and assuming that the light quark acts as a *spectator* (i.e. non interacting), estimate the lifetimes of the D^+ and D^0 mesons, and compare in each case with the tabulated values of the PDG. The *branching fraction* for a given final state of a meson $A \to i$ is defined as the ratio Γ_i/Γ_A of partial width to total width. Estimate the semileptonic (lepton+X) branching fractions, into electron and muon, and compare with the PDG data for these two mesons.

c) Estimate the *purely leptonic* decay fraction of the pseudoscalar mesons D^+ and D_s^+ into $\mu^+\nu_\mu$, taking into account their chiral suppression (as for the pion), and also compare with PDG data. Use for that purpose $f_{D^+} = 210 MeV$ and $f_{D_s^+} = 250 MeV$.

Problem 5 ($\star\star$)

Bottom quark mean lifetime

Try to understand the mean lifetime of the b-quark in the range of the ps, as function of its mass. Like the charm, the b-quark decays into other (lighter) fermions by means of left-handed charged currents in the $V-A$ theory, and the muon decay should be taken as a model. Write down the main 14 modes in which the b-quark can decay, and neglect *squared* masses of all final-state fermions. Notice that now, b-quark decay spans over the *three* known generations.

Break down the total b-quark decay width into two partial widths, one that includes (e, μ, τ) leptons ($\Gamma_l = \Gamma_e + \Gamma_\mu + \Gamma_\tau$) and another including only hadrons: $\Gamma_b = \Gamma_l + \Gamma_{had}$. Using the Cabibbo angle ($\lambda \equiv \sin\theta_c$), and a color factor 3 for hadronic decays, show that both Γ_l and Γ_{had} contain a factor $|V_{cb}|^2 + |V_{ub}|^2$. Calculate the b-quark lifetime in *two* different scenarios for the quark mixing matrix (CKM):

a) λ is "flavor independent", thus having $|V_{cb}| = \lambda$ and $|V_{ub}| = 0$.

b) Wolfenstein's conjecture is verified, with: $|V_{cb}| \sim \lambda^2$ and $|V_{ub}| \sim \lambda^3$.

Comment on the agreement between each of the above scenarios and the data from the SLAC experiments in 1983 facilitated in the course, where the mean b-quark lifetime was first measured. Also compare with current data, specifically refering to B^+, B^0, and B_s^0 meson lifetimes. Are they close enough to each other as to justify the hypothesis of a b-quark lifetime, within the meson? Would the agreement become better by reducing the available phase space for c and τ?

Now perform a specific prediction for the b-quark branching fraction into electron and muon. Do they agree well with recent data? Notice that the PDG offers inclusive b-quark data, and not only meson data.

Finally, estimate the decay fraction of the $B^+[u\bar{b}]$ meson into $\mu^+\nu_\mu$ and $e^+\nu_e$, using the formula seen in Lecture IV for the pion, with decay constant $f_B = 190 MeV$ and B^+-meson mass.

Problem 6 ($\star\star$)

Top quark mean lifetime

a) Calculate the mean lifetime of the t-quark (in s), with mass $m_t = 174\,\mathrm{GeV/c^2}$, in the Standard Model. Take into account that since $m_t > m_b + m_W$, the top quark can now decay into a *real* W, with mass $m_W = 80.4\,\mathrm{GeV/c^2}$, $t \to bW^+$, and not just a *virtual* one, as in the previous cases. Assuming an unpolarized t-quark, perform the detailed calculation of the spin sum in the Feynman diagram, using the completeness relation for spin 1 massive bosons $\sum_{\lambda=\pm1,0} \epsilon_\mu^\lambda {\epsilon_\nu}^{\lambda *} = -g_{\mu\nu} + p_\mu p_\nu/M^2$, as given in the statement of Exercise 21. The completeness relation means summing over the *physical* polarization states of the vector boson (2 transverse, with $S_z = \pm1$, and one longitudinal, with $S_z = 0$).
In the *unitary* gauge, the above diagram is the only one possible, to lowest order [17].
Proceed to cancel the tensors with opposite symmetry, and assess the final state 4-vectors in the t-quark rest frame. Justify that the b-quark mass can now be neglected. The generic two-body partial decay width formula, that is given in Problem 17, must also be used.

b) From the result obtained, provide a reasoned answer to the question: can the t-quark form mesons or baryons, through the strong interaction in QCD?

[17] it is interesting to note that in other gauges, $t \to b\phi^+$ also contributes. But then extra, *unphysical* polarization states of the W^+ need to be included in the sum, yielding for instance: $\sum_\lambda \epsilon_\mu^\lambda {\epsilon_\nu}^{\lambda *} = -g_{\mu\nu}$. Remarkably, an *identical* result is obtained for the mean t-quark lifetime when the $t \to b\phi^+$ and $t \to bW^+$ diagrams are summed, as it was to be expected.

Problem 7 ($\star$)

Neutral meson oscillation

Assuming CPT conservation, the neutral mesons P^0 and $\bar{P}^0$ have equal mass m_0 and width Γ_0. Their most general weak interaction Hamiltonian is: $H \equiv \mathcal{M} - \frac{i}{2}\mathit{\Gamma}$, $\mathcal{M} = \begin{pmatrix} m_0 & |M_{12}| \\ |M_{12}| & m_0 \end{pmatrix}$, $\mathit{\Gamma} = \begin{pmatrix} \Gamma_0 & |\Gamma_{12}|e^{i\omega} \\ |\Gamma_{12}|e^{-i\omega} & \Gamma_0 \end{pmatrix}$ with $\mathcal{M}$ and $\mathit{\Gamma}$ *Hermitian*, $\omega \neq 0$ signaling CP-violation. H can be *diagonalized* by a *similarity* transformation:

$$\begin{pmatrix} \mu_H & 0 \\ 0 & \mu_L \end{pmatrix} = \begin{pmatrix} M_H - \frac{i}{2}\Gamma_H & 0 \\ 0 & M_L - \frac{i}{2}\Gamma_L \end{pmatrix} = X^{-1}\left(\mathcal{M} - \frac{i}{2}\mathit{\Gamma}\right)X = \frac{1}{2pq}\begin{pmatrix} q & q \\ p & -p \end{pmatrix}\left(\mathcal{M} - \frac{i}{2}\mathit{\Gamma}\right)\begin{pmatrix} p & q \\ p & -q \end{pmatrix}$$

with $p, q \in \mathbb{C}$ and $|p|^2 + |q|^2 = 1$. The eigenvectors (eigenstates): $|P_{H,L}\rangle = p|P^0\rangle \pm q|\bar{P}^0\rangle$ acquire *different* masses and widths: $\Delta M \equiv M_H - M_L > 0$ and $\Delta\Gamma \equiv \Gamma_H - \Gamma_L$, arising from the weak interaction above. Using Schrödinger's equation, show that the time evolution of the $|P^0(t)\rangle$ and $|\bar{P}^0(t)\rangle$ states, *which were tagged at $t = 0$ as P^0 and $\bar{P}^0$*, respectively, is given by:

$$\begin{pmatrix} |P^0(t)\rangle \\ |\bar{P}^0(t)\rangle \end{pmatrix} = \begin{pmatrix} g_+(t) & \frac{q}{p}g_-(t) \\ \frac{p}{q}g_-(t) & g_+(t) \end{pmatrix}\begin{pmatrix} |P^0\rangle \\ |\bar{P}^0\rangle \end{pmatrix} \tag{1}$$

with the functions: $g_\pm(t) = \frac{1}{2}\left(e^{-\Gamma_H t/2}e^{-iM_H t} \pm e^{-\Gamma_L t/2}e^{-iM_L t}\right)$, and that these verify ($\Gamma \equiv \frac{\Gamma_H + \Gamma_L}{2}$):

$$\begin{aligned} |g_\pm(t)|^2 &= \frac{1}{2}e^{-\Gamma t}[\cosh(\Delta\Gamma t/2) \pm \cos(\Delta M t)] \\ g_+^*(t)g_-(t) &= -\frac{1}{2}e^{-\Gamma t}[\sinh(\Delta\Gamma t/2) + i\sin(\Delta M t)] \end{aligned} \tag{2}$$

For that purpose take into account the property: $e^{X^{-1}H_D X} = X^{-1}e^{H_D}X$. This result is needed to solve Problem 16, and may be applied to all cases of meson oscillation that are refered to in Problem 15.
CP-violation is *observable* in the oscillations, through: $\delta = |p|^2 - |q|^2 = (|H_{12}| - |H_{21}|)/(|H_{12}| + |H_{21}|) \neq 0$

Problem 8 ($\star\star$)

V+A part of neutral current

The detailed analysis of the differential scattering cross-section of muonic neutrinos and antineutrinos, off an isoscalar target (marble, Fe), as function of the y variable of Bjorken, allowed to establish the *not entirely chiral* nature of the neutral current, in a direct way. With the same isoscalar target, charged current events were used to determine the partonic densities. Using the neutral current events, the left-handed (L) and right-handed (R) chiral couplings were defined: $g^{\text{iso}}_{R,L}$. For each type of quark i, within the proton and neutron, the definition reads as follows: $g^i_{R,L} = \frac{1}{2}(C^i_V \mp C^i_A)$. What is measured for the isoscalar target corresponds to the quadratic average of u- and d-quarks, the s-quark contribution being almost zero for $x \gtrsim 0.1$. The experimental result is summarized as: $\left(g^{\text{iso}}_L\right)^2 = 0.300 \pm 0.015$ and $\left(g^{\text{iso}}_R\right)^2 = 0.024 \pm 0.008$.

a) Find out the prediction of the SU(2)$_L \times$U(1)$_Y$ theory for these two parameters, and compare them with the data.

b) Determine, by small variations of $\sin^2\theta_W$, its best-fit result from both numbers.

These experiments were described in the course.

Problem 9 ($\star\star$)

Antimatter fraction in the proton

Show that the total fraction of the proton (or neutron) momentum carried by the antiquarks can be calculated as: $r \equiv \int x\bar{Q}(x)dx / \int xQ(x)dx = (3R-1)/(3-R)$, with $R \equiv \sigma(\bar{\nu}_\mu)/\sigma(\nu_\mu)$ being the antineutrino/neutrino ratio of total cross-sections measured on an isoscalar target, where $Q(x) \equiv d(x) + u(x)$ and $\bar{Q}(x) \equiv \bar{u}(x) + \bar{d}(x)$ are the generic partonic densities seen in the course. The Bjorken variable x indicates the fraction of the proton or neutron momentum. Ignore the s quark contribution in the derivation of the above formula. Which quark is created with highest probability when an s quark is struck by a neutrino or antineutrino? Indicate specific final states that could be taken as a proof for the presence of s quarks in the proton, using neutrino beams. Can R be experimentally determined with the exclusion of the s quark?

Problem 10 ($\star\star$)

Neutral current fraction

Using the SU(2)$_L \times$U(1)$_Y$ theory of the Standard Model, show the dependence on $\sin^2\theta_W \equiv x_w$ of the ratios between the total cross-section of neutral currents and charged currents, for neutrino scattering on an isoscalar target N, for $x \gtrsim 0.1$. For simplicity, the antiquark density can be neglected, and only u and d quarks can be used. The expressions to be proven read as follows:

$$R_\nu = \frac{\sigma(\nu_\mu N \to \nu_\mu X)}{\sigma(\nu_\mu N \to \mu^- X)} = \frac{1}{2} - x_w + \frac{20x_w^2}{27} \qquad R_{\bar\nu} = \frac{\sigma(\bar\nu_\mu N \to \bar\nu_\mu X)}{\sigma(\bar\nu_\mu N \to \mu^+ X)} = \frac{1}{2} - x_w + \frac{20x_w^2}{9}$$

Obtain the best fit to $\sin^2\theta_W$ from the final data on neutral currents by neutrino experiments in the 1980 decade, that were reported in the course: $R_\nu = 0.33 \pm 0.01$ and $R_{\bar\nu} = 0.38 \pm 0.02$.

Problem 11 ($\star\star$)

Discrete symmetries and CKM matrix

a)The $K^0_{S,L}$ are mass eigenstates of the K^0 meson that do not turn out to be perfect CP eigenstates, but contain some contamination ϵ from the opposite eigenvalue, in the form: $|K^0_{S,L}\rangle = 1/\sqrt{N}\,(|K_{1,2}\rangle + \epsilon|K_{2,1}\rangle)$, where $|K_{1,2}\rangle$ are the perfect CP eigenstates: $|K^0_{1,2}\rangle \equiv 1/\sqrt{2}\,(|K^0\rangle \pm |\bar{K}^0\rangle)$. This contamination was determined by the historical experiment of Cronin and Fitch in 1964, who obtained the value: $|\epsilon| = 2.23(1) \times 10^{-3}$. To that end, the $K^0_L \to \pi^+\pi^-$ decay fraction was used.

It is easily understood that, as we have seen in β-decay, the V$-$A theory may also produce the semileptonic decay: $K^0_L \to \pi^- e^+ \nu_e$, as well as its charge conjugate: $\bar{K}^0_L \to \pi^+ e^- \bar{\nu}_e$. In fact, semileptonic decays represent 41 % of the total decays of K^0_L. Draw a lowest order Feynman diagram that can represent each of them. In the absence of phases in the CKM matrix, may these diagrams explain an observable difference between the decay rates of the above charged conjugate modes? Explain the answer, according to the statement of Exercise 23. Mention explicitely the chirality of the particles involved, indicating why CP-symmetry comes into play.

b) The result of Cronin-Fitch suggests to perform a precision measurement of the previously mentioned asymmetry. Do you think that the electron/positron identification can be used to *tag* the K^0_L content of $\bar{K}^0/K^0$? Assume that the following asymmetry has been measured:

$$\delta_{SL} \equiv \frac{\Gamma(K^0_L \to \pi^- e^+ \nu_e) - \Gamma(\bar{K}^0_L \to \pi^+ e^- \bar{\nu}_e)}{\Gamma(K^0_L \to \pi^- e^+ \nu_e) + \Gamma(\bar{K}^0_L \to \pi^+ e^- \bar{\nu}_e)}$$

as it was indeed done for the first time in 1974 (S. Gjesdal et al.), with the result: $\delta_{SL} = 3.27(12) \times 10^{-3}$ (current value). Can you assess the compatibility or not between this result and that of Cronin-Fitch? If, while doing this, you find that the ϵ constant must be a complex number, then determine its phase in degrees. You must take into account the relativistic expression of a partial decay width from its amplitude $\mathcal{M}$, and the fact that $\bar{K}^0 \not\to \pi^- e^+ \nu_e$.

c) Draw a box Feynman diagram, with two W bosons in a loop, that is able to convert the K^0 into its antiparticle $\bar{K}^0$. May the presence of phases in the CKM matrix be responsible of the $|\epsilon| \neq 0$ value in the $K^0_{S,L}$ mass eigenstates? Provide only a qualitative explanation.

Problema 12 ($\star\star$)

The Weinberg angle

In the SU(2)$_L\times$U(1)$_Y$ theory, the neutral spin 1 gauge bosons W^3_μ and B_μ interact with the fermionic currents $J^{3,\mu}$ and $J^{Y,\mu}$ according to the lagrangian density:

$$\mathcal{L}_{\text{neutral}} = \begin{pmatrix} J^{3,\mu}, & \frac{1}{2}J^{Y,\mu} \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix} \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix}$$

where $J^{Y,\mu} \equiv 2(J^{em,\mu} - J^{3,\mu})$. Weinberg postulates that, due to the Higgs mechanism, the physical bosons Z_μ and A_μ are orthogonal linear combinations of the gauge bosons, according to the rotation:

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix}$$

with $c \equiv \cos(\theta_W)$, $s \equiv \sin(\theta_W)$, and that the above density may be written as:

$$\mathcal{L}_{\text{neutral}} = \begin{pmatrix} J^{NC,\mu}, & eJ^{em,\mu} \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}$$

where $J^{NC,\mu}$ is the *physical* neutral current that couples to the Z^0 and $J^{em,\mu}$ is the electromagnetic current that couples to the photon, with e being the magnitude of the electron charge. Show that there is a unique solution to Weinberg's proposal, given by:

a) the double equation: $e = sg = cg'$.

b) the neutral current in the form: $J^{NC,\mu} = \frac{g}{c}(J^{3,\mu} - s^2 J^{em,\mu})$.

Is a V+A coupling of the neutral current expected in this model? Why? Simply analyse the 2×2 linear system that arises from equating the multipying factors to W^3_μ and B_μ in both equations for $\mathcal{L}_{\text{neutral}}$, taking e and $J^{NC,\mu}$ as unknowns. The above equations are the core of the electroweak unification theory.

Problema 13 ($\star\star\star$)

Three neutrino families

a) Use the result of Exercise 21, in the framework of the electroweak unification theory, to show that the partial decay width of the Z^0 into $\nu_\mu \bar{\nu}_\mu$ is calculated to be

$$\Gamma(Z^0 \to \nu_\mu \bar{\nu}_\mu) = \frac{G_F}{12\pi\sqrt{2}} M_Z^3$$

and that it takes the value 166 MeV from the above constants. Make clear whether only the $SU(2)_L \times U(1)_Y$ theory has been used to achieve the previous prediction, or scalar fields need to be added to the theory, with a given hypercharge value, according to the Higgs mechanism.

b) Calculate the partial decay widths of the Z^0 (in MeV) into neutrinos, hadrons and charged leptons. To that end, consider the C_V and C_A couplings of the neutral current for each type of fermion, take into account the factor 3 of color, and a precise value of $\sin^2\theta_W$. Also note that the t-quark is not reachable from the Z^0. In addition calculate the total decay width of the Z^0 and its mean lifetime in s.

c) Assuming that the confusion between hadronic and leptonic events is very small in a e^+e^- collider, with energy $\sqrt{s} = M_Z$, explain how could you precisely determine $\Gamma_{\nu\bar{\nu}}$, and whether by doing so, you could find out how many light neutrino families exist. Use here the Z^0 resonant production cross-section: $\sigma(e^+e^- \to Z^0 \to f\bar{f}) = 12\pi \left(\frac{\Gamma_e \Gamma_f}{M_Z^2}\right) \frac{s}{(s - M_z^2)^2 + M_Z^2 \Gamma_Z^2}$, as it is shown in Exercise 22. ¿What is the $\sigma(e^+e^- \to Z^0 \to \mu^+\mu^-)$ value in nb on the peak? Do the theoretical and experimental values exactly coincide? The above program was carried out in 1991 by the 4 experiments at the LEP collider at CERN. Show that you have understood the method, from the data of the L3 experiment facilitated in the course. Could $\Gamma_{\nu\bar{\nu}}$ also be determined in a direct way, by observing the bremsstrahlung photon? Explain why the Z^0 line-shapes appear to be equal for all leptons, and for hadrons.

Problem 14 ($\star\star\star$)

Asymmetry in electron-positron annihilation into muon pairs

For a sufficiently high collider energy ($\sqrt{s} = \sqrt{q^2}$), the electroweak contribution to $e^+e^- \to \mu^+\mu^-$ becomes significant, and even exceeds, the electromagnetic contribution. But, more importantly, quantum interference takes place between the Z^0 and the photon, that is very sensitive to the parameters of electroweak unification. As has been seen, their respective amplitudes are:

$$\mathcal{M}_\gamma = -\frac{e^2}{q^2}\left(\bar{\mu}\gamma^\mu \mu\right)\left(\bar{e}\gamma_\mu e\right)$$

$$\mathcal{M}_Z = -\frac{g^2}{4\cos^2\theta_W}\left[\bar{\mu}\gamma^\nu (C_V^\mu - C_A^\mu \gamma^5)\mu)\right]\left(\frac{g_{\nu\sigma} - q_\nu q_\sigma / M_Z^2}{q^2 - M_Z^2}\right)\left[\bar{e}\gamma^\sigma (C_V^e - C_A^e\gamma^5)e)\right]$$

where we have assumed $\rho = 1$ according to the Higgs mechanism. We have denoted the muon and electron spinors by the symbols (μ, e), reserving (ν, σ) for the relativistic indices. Taking into account the *helicity conservation* (see Exercise 18), we may isolate in $\mathcal{M}_Z$ the only two relevant chiral components (R and L) for *each* fermion, as is evident from expression: $C_V - C_A\gamma^5 = \frac{1}{2}(C_V - C_A)(1+\gamma^5) + \frac{1}{2}(C_V + C_A)(1-\gamma^5)$, with the respective coefficients: $g_R \equiv C_V - C_A$ y $g_L \equiv C_V + C_A$. The helicity conservation happens in the ultrarelativistic limit of high collision energy ($\sqrt{s} \gg 2m_\mu$). Check that the second term in the numerator of the propagator does not contribute, using Dirac's equation for μ and e.
Hence the result can be rewritten as:

$$\mathcal{M}_Z = -\frac{\sqrt{2}G_F M_Z^2}{s - M_Z^2}\left[g_R^\mu\left(\bar{\mu}_R\gamma^\nu\mu_R\right) + g_L^\mu\left(\bar{\mu}_L\gamma^\nu\mu_L\right)\right]\left[g_R^e\left(\bar{e}_R\gamma_\nu e_R\right) + g_L^e\left(\bar{e}_L\gamma_\nu e_L\right)\right]$$

Problem 14 ($\star\star\star$)

Asymmetry in electron-positron annihilation into muon pairs (continued)

a) In the expression $|\mathcal{M}_\gamma + \mathcal{M}_Z|^2$, isolate each of the 4 helicity contributions to the differential cross-section in the center-of-mass frame (CM), as indicated in Exercises 18 and 22. Show that the first of them takes the form:

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM}\left(e_L^+ e_R^- \to \mu_L^+ \mu_R^-\right) = \frac{\alpha^2}{4s}(1+\cos\theta)^2\left|1 + r g_R^\mu g_R^e\right|^2$$

and analogously obtain the others, with $r \equiv (\sqrt{2}G_F M_Z^2 s/e^2)/(s - M_Z^2 + iM_Z\Gamma_Z)$. The imaginary part at the pole of the propagator is the way to take into account the total width Γ_Z of the Z^0, in a relativistic way, and $\alpha \equiv e^2/(4\pi\epsilon_0\hbar c) = e^2/4\pi$ is the fine structure constant.

b) Show that, as a result of the $\gamma - Z^0$ interference, the angular distribution in the CM frame is:

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM}\left(e^+ e^- \to \mu^+ \mu^-\right) = \frac{\alpha^2}{4s}\left[A_0\left(1+\cos^2\theta\right) + A_1\cos\theta\right]$$

with the emergence of an asymmetric component proportional to A_1. Determine A_0 and A_1 in the Standard Model at lowest order, as function of $\mathrm{Re}(r)$ and $|r|^2$. Draw the very different behaviour that $\mathrm{Re}(r)$ and $|r|^2$ have, as function of the energy $\sqrt{s}$, in the neighbourhood, and far away, from the Z^0 mass. The hypothesis of universality implies equal couplings: $g^\mu_{R,L} = g^e_{R,L} \equiv g_{R,L}$.

c) Define $A_{FB} \equiv \frac{F-B}{F+B}$, with $F \equiv \int_0^1 (d\sigma/d\Omega)d\Omega$ and $B \equiv \int_{-1}^0 (d\sigma/d\Omega)d\Omega$, as the *charge asymmetry* refered to the incoming e^+ and the outgoing μ^+, and calculate it in the Standard Model, to lowest order. Determine its numerical value (in %) and *sign* for a collision energy $\sqrt{s} = 34.6\,\mathrm{GeV}$, that lies below M_Z ($s \ll M_Z^2$), but much above the vector resonances associated to the 5 light and heavy quarks: u, d, s, c and b. This result played a decisive role in the selection of the electroweak unification theory of Weinberg-Salam, based on the Higgs doublet with hypercharge $+1$ ($\rho = 1$), when the above asymmetry was measured with 1 % precision at the PETRA accelerator (Hamburg) in 1983. See the data of the Mark J experiment indicated in the course.

Problem 15 ($\star\star$)

Oscillation, CKM matrix, and CP violation

a) Justify that, in the Standard Model (where neutrinos do not have mass), the only particles that can spontaneously convert into their antiparticles, and then oscillate, are the *neutral mesons that live long enough*. Explain why there are only *four* neutral mesons of this type, and indicate their quark content. Draw, for each of them, a Feynman diagram with *4 virtual particles* that may trigger such spontaneous transition, through a quantum loop. Why the $K^{*0}(892)$ meson cannot oscillate? Can the neutron do it? Is CP violation a necessary consequence of the oscillation process?

b) As you know, in the SM a down-type quark cannot convert itself, in a *direct* way, into another down-type quark of different flavor, by emitting a photon, or a Z^0, or a gluon g. However, such coupling may occur in an *indirect* way through vacuum polarization diagrams with a loop (so called *penguin* diagrams): a virtual W is emitted and reabsorbed between the two quarks of different flavor, the intermediate quark being coupled to a gluon g, or to a Z^0. Draw at least 3 diagrams of this kind.
Draw a *penguin* diagram to explain the decay $\bar{B}_s^0 \to K^+\pi^-$, and also a competing *tree* diagram that achieves the same goal (*tree* diagram just means absence of loops). You may let the $\bar{s}$-quark be a spectator, in both cases. Explain which observable you would choose experimentally, in order to show the non conservation of CP symmetry, in this particular decay. Why do you need *two* amplitudes, instead of only one? Which elements of the CKM matrix would be decisive?

c) If you were to communicate to someone at a distant galaxy, that may be entirely formed by antimatter, that the Earth rotates around the Sun counterclockwise in the solar system (from the north pole of the Sun), how would you do it? Take into account that you need to define some common reference about the helicity *sign*, and, without knowing if they live in an antiparticle world or not, there is no way to define the *sign* of the electric charge. Would it help using, for this purpose, the semileptonic decay $K_L^0 \to \pi^- e^+ \nu_e$, dealt with in Problem 11? Indicate what the message would contain. See another version of this problem in Griffiths' book, Chapter 4 (p. 114).

Problem 16 ($\star\star$)

CP-asymmetry in oscillation

Using the result of Problem 7 for the time evolution of the meson wave function projections $|P^0\rangle$ and $|\bar{P}^0\rangle$ (formulae (1) and (2)):

a) Determine the probability density, as function of time (measured at the instant of decay t), that the meson tagged at $t=0$ as $|P^0\rangle$:

- has remained $|P^0\rangle$
- has jumped to $|\bar{P}^0\rangle$

b) Show that the particle/antiparticle rate asymmetry, observed at $t>0$ through the decays $B_s \to D_s^- X \mu^+ \nu_\mu$ and $\bar{B}_s \to D_s^+ X \mu^- \bar{\nu}_\mu$, where X is a set of hadrons, depends on t as:

$$a_{SL}^s = \frac{N\left(B_s\right) - N\left(\bar{B}_s\right)}{N\left(B_s\right) + N\left(\bar{B}_s\right)} = \delta \cdot \left(1 - \frac{\cos(\Delta M\, t)}{\cosh(\Delta\Gamma t/2)}\right) \quad \text{with} \quad \delta \equiv |p|^2 - |q|^2 \ll 1$$

Assume that the accelerator collisions produce b and $\bar{b}$ quarks with equal probability at $t=0$. Note it can shown that $\delta \neq 0$ indicates violation of the CP-symmetry in the oscillation process, as $|H_{12}| \neq |H_{21}|$ in the mixing matrix of Problem 7. The above asymmetries for the B_s meson are currently under study at experiments with high time resolution, as LHCb ($50 fs$).

Problem 17 ($\star\star$)

Mean lifetime of the Higgs boson

a) Show that, in the Standard Model, the partial decay width of the Higgs boson into a given fermion-antifermion pair $f\bar{f}$ is given by:

$$\Gamma(H^0 \to f\bar{f}) = \frac{G_F M_H m_f^2}{4\pi\sqrt{2}}$$

For that purpose, use the lagrangian coupling of the $H^0 f\bar{f}$ vertex seen in the course, and perform the direct calculation of the Feynman diagram, by applying Casimir's rule for the spin sum. Be reminded of the generic result for the partial width of a relativistic two-body decay process of a particle $A \to 1+2$: $\Gamma(A \to 1+2) = \frac{|\boldsymbol{p}|}{32\pi^2 m_A^2}\int \overline{|\mathcal{M}|^2} d\Omega$ where $|\boldsymbol{p}|$ stands for the common momentum in its rest frame.

b) Assess numerically in MeV the total Higgs width into quarks and leptons, using its mass $M_H = 125\,\mathrm{GeV}/c^2$, and the specific prediction for $b\bar{b}$ pairs ($\Gamma_{b\bar{b}}$). Is $\Gamma(H^0 \to \gamma\gamma)$ comparatively relevant? Provide a reasoned answer, assuming the two photons are produced through a triangular loop involving the t-quark.

c) Neglecting the contribution of W^+W^- and Z^0Z^0 pairs to the total width (below their mass shell), provide an approximate upper bound to the Higgs boson lifetime (in s). Discuss whether it is long enough as to produce visible secondary vertices, at the LHC [18].The Higgs boson is produced through a t-quark loop in the reverse process $gg \to H^0$, where each gluon comes from a colliding proton. In order to estimate the H^0 momentum, assume a maximum difference between the Bjorken x of the gluons in the collision is of order 0.1, similar to the one observed with neutrinos for the antiquark distribution in the proton (Lecture V).

The above results for $\Gamma(H^0 \to f\bar{f}\ )$ played a decisive role in the interpretation of the Higgs boson discovered at the LHC by ATLAS and CMS in 2012.

[18] semiconductor microvertex detectors cannot reach precisions signiticantly better than $10\,\mu m$, on individual particle hits.

AUXILIARY EXERCISES

General formalism that helps assessing the problems

Exercise 18 ($\star$)

Helicity conservation

a) Show that, in the ultrarelativistic limit ($E \gg m$), every external fermionic vertex of a diagram in a *vector* theory $\bar{u}\gamma^\mu u$ only couples spinors of the same helicity, since $\bar{u}_L\gamma^\mu u_R = \bar{u}_R\gamma^\mu u_L = 0$. And that the same result applies for an axial-vector theory $\bar{u}\gamma^5\gamma^\mu u$. It is called *helicity conservation* in the literature, and plays an important role in QED, in the electroweak theory, as well as in QCD.

b) In particular, for the emblematic process in QED $e^+e^- \to \mu^+\mu^-$, to order α^2, express the amplitude $\mathcal{M}_\gamma = -(e^2/q^2)(\bar{\mu}\gamma^\mu\mu)(\bar{e}\gamma_\mu e)$ as a sum of its chiral components, and *draw* the incoming and outgoing momenta, the spin orientations and the scattering angle in the CM frame, for each of the *4 possible helicity configurations*.

c) The invariance under spatial rotations of the above amplitude has an interesting consequence: these 4 amplitudes must be proportional to the elements of the Wigner matrices: $d^j_{\lambda'\lambda}(\theta) = \langle j\lambda'|e^{i\theta J_y}|j\lambda\rangle$, where $e^{i\theta J_y}$ represents a rotation of angle θ about an axis $\perp$ to the reaction plane. Check in the tables that:

$$d^1_{11}(\theta) = d^1_{-1-1}(\theta) = \frac{1}{2}(1+\cos\theta) = -u/s$$

$$d^1_{1-1}(\theta) = d^1_{-11}(\theta) = \frac{1}{2}(1-\cos\theta) = -t/s$$

where the last equality happens in the ultrarelativistic limit. The above allows to derive, only using helicity amplitudes, the angular distribution obtained in QED (α^2) for $e^+e^- \to \mu^+\mu^-$, as the quadratic sum of the two amplitudes ($\overline{|\mathcal{M}|^2} = e^4[2(t/s)^2 + 2(u/s)^2]$), with $e^2 = 4\pi\alpha$, thus:

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM} = \frac{1}{64\pi^2 s}\overline{|\mathcal{M}|^2} = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right) = \frac{\alpha^2}{4s}\frac{1}{2}\left[(1+\cos\theta)^2 + (1-\cos\theta)^2\right]$$

Check in the previous spin drawings that each amplitude has the characteristic backward helicity suppression, imposed by angular momentum conservation (similar case to the neutrinos). However, the spin sum is unsuppressed, the angular distribution of the outgoing μ^+ with respect to the incoming e^+ being *symmetric* ($\cos\theta \to -\cos\theta$). This exercise helps understanding Problem 14.

Exercise 19 ($\star$)

The polarization 4-vector

The most general quantum state of a fermion of mass m is characterized by the mean value of its spin $\langle \boldsymbol{S} \rangle$ in its rest frame, and by its 3-momentum $\boldsymbol{p}$ (4-vector p^μ) [19].The spin also admits a 4-vector representation. Indeed, we may define, in the rest frame, the 4-vector: $s^\mu \equiv (0, 2\langle \boldsymbol{S} \rangle)$, with $s \cdot p = 0$ and $s^2 = -1$, and then apply a Lorentz transformation (boost) in any desired direction. Hence a fermion spinor may be denoted as $u(p,s)$ ($v(p,s)$ for the antifermion) whenever it is required. The 3-vectors $\boldsymbol{p}$ and $\langle \boldsymbol{S} \rangle$ form an angle θ that is actually a constant of the free relativistic motion: $2\langle \boldsymbol{S} \rangle \cdot \boldsymbol{p}/|\boldsymbol{p}| \equiv \cos\theta$, as it can be shown.

a) Show that the helicity states $\lambda = \pm 1$ of the fermion correspond to the 4-vectors:

$$s^\mu = \lambda(|\boldsymbol{p}|, 0, 0, E)/m \qquad \text{with } m \neq 0$$

b) Show that for $\boldsymbol{p} = (p\sin\theta, 0, p\cos\theta)$ the spinor of helicity $\lambda = +1$ takes the form:

$$u(p,s) = N(\cos(\theta/2),\ \sin(\theta/2), k\cos(\theta/2), k\sin(\theta/2))$$

with the constants $k = |\boldsymbol{p}|/(E+m)$ and $N = \sqrt{E+m}$. To do that, make a rotation of angle θ about the X-axis ($e^{-i\sigma_2\theta/2}$) on both components of the reference spinor.

[19] the relativistic spin operator takes the form: $\boldsymbol{S} \equiv (1/2)\,\mathrm{diag}\,(\boldsymbol{\sigma}, \boldsymbol{\sigma})$.

Exercise 20 ($\star\star$)

The polarized spin projectors

Designating as $u_\lambda(p,s)$ the fermion states with polarization 4-vector λs^μ, with $s^\mu = (|\boldsymbol{p}|, 0, 0, E)/m$, and $\lambda = \pm 1$, it is evident that the 4 states: $\{u_\lambda(p,s), v_\lambda(p,s), \lambda = \pm 1\}$, where $v_\lambda = \gamma^5 u_{-\lambda}$ corresponds to the antifermion, form an orthonormal basis within the Hilbert space of dimension 4 of its quantum states with given three-momentum $\boldsymbol{p}$.
That the 4×4 *matrices*: $\rho^u_\lambda = u_\lambda(p,s)\bar{u}_\lambda(p,s)$ and $\rho^v_\lambda = v_\lambda(p,s)\bar{v}_\lambda(p,s)$ are the projectors, in the above Hilbert space, over the respective states $u_\lambda(p,s)$ and $v_\lambda(p,s)$, with $\lambda = \pm 1$, becomes evident, from their own definition.

a) Show that the expression *without sum* is verified: $u_\lambda \bar{u}_\lambda = \frac{1}{2}(m + \not{p})(1 + \lambda\gamma^5 \not{s})$, with $\lambda = \pm 1$.
b) Then show that: $-v_\lambda \bar{v}_\lambda = \frac{1}{2}(m - \not{p})(1 + \lambda\gamma^5 \not{s})$.

Note that it suffices to perform the proof *in the fermion rest frame*, since the Lorentz invariance of the operators is manifest, as they are formulated with *slashed* 4-vectors.

It is suggested to show that the given expressions without sum $u_\lambda \bar{u}_\lambda$ are nothing but the projectors ρ^u_λ defined above (that is, they either leave the 4 aforementioned base states invariant, or throw zero). Alternatively, you can prove the expressions directly, as 4×4 matrices in the fermion rest frame, by using the explicit form of the u_λ and v_λ spinors from Lecture I.

The above expressions allow the assessment of amplitudes of *Feynman diagrams with external polarized fermions*. The method consists in including new factors in the trace calculations, when applying Casimir's rule. In particular, this result is key to solve Problem 2.

Exercise 21 ($\star\star$)

Width of a vector boson

Assume that a spin 1 boson A of mass M_A decays into two fermions f_1 and $\bar{f}_2$ of spin 1/2 with a lagrangian coupling given by: $\Gamma^\mu \equiv -ig_X\gamma^\mu \frac{1}{2}(C_V - C_A\gamma^5)$, with $m_{f_i} \ll M_A$, and $C_{V,A} \in \mathbb{R}$. In absence of polarization, no angle is relevant in the center-of-mass frame. Show that the partial width is given by:

$$\Gamma(X \to f_1\bar{f}_2) = \frac{g_X^2}{48\pi}(C_V^2 + C_A^2)M_A \ .$$

Start from the amplitude: $\mathcal{M} = \epsilon_\mu^\lambda\,(\bar{u}(k')\Gamma^\mu u(k))$, where ϵ_μ^λ is the 4-vector of polarization λ of the spin 1 boson. Then use the ***completeness relation***: $\sum_{\lambda=0,\pm1}\epsilon_\mu^\lambda\epsilon_\nu{}^{\lambda*} = -g_{\mu\nu} + p_\mu p_\nu/M_A^2$, with $A_\mu = \epsilon_\mu^\lambda e^{-ip\cdot x}$ being the general solution to the differential equation $(\Box + M_A^2)A_\mu = 0$, where the condition $\partial_\mu A^\mu = 0$ *must* be fulfilled ($M_A \neq 0$). Therefore $\epsilon^\lambda \cdot p = 0$, and we have 3 independent modes of oscillation $\lambda = 0, \pm1$ defined by the 4-vector ϵ_μ^λ, with the normalization: $|\epsilon^\lambda|^2 = -1\ \forall\lambda$. Note the analogy with the s_μ 4-vector for the fermions.

Next show that: $\sum_\lambda \mathcal{M}\mathcal{M}^* = (g_X^2/4)\left(\sum_\lambda \epsilon_\mu^\lambda\epsilon_\nu^{\lambda*}/3\right)\left[\left(C_V^2 + C_A^2\right)T_1^{\mu\nu} - 2C_VC_AT_2^{\mu\nu}\right]$ is the spin averaged squared amplitude, after identifying the tensors $T_{1,2}^{\mu\nu}$, on application of Casimir's rule. The generic result of the partial decay width for a two-body process, indicated in Problem 17, should also be used. Assess the 4-vectors of the particles involved in the boson rest frame.

This result has a direct application to the Z^0 boson decays (in particular to Problem 13), as well as to $W^\pm$,$\Upsilon[b\bar{b}]$ and $J/\psi[c\bar{c}]$ decays, and can be extended to the polarized case.

Exercise 22 ($\star\star\star$)

Line-shape of the Z^0 boson

Show that $\sigma(e^+e^- \to Z^0 \to f\bar{f}) = 12\pi \left(\frac{\Gamma_e \Gamma_f}{M_Z^2}\right) \frac{s}{(s-M_z^2)^2 + M_Z^2\Gamma_Z^2}$ is the e^+e^- annihilation cross-section that corresponds to the amplitude $\mathcal{M}_Z$ of Problem 14, for $\sqrt{s} = 2E_B$ in the neighbourhood of M_Z, and f being a particular family of quarks or leptons. Follow the steps below:

a) Start from the 4 *chiral* components of $\mathcal{M}_Z$ that are explicitated in Problem 14.

b) Apply the results of Exercise 18 for the photon, replacing $\frac{e^2}{s}$ by $g_z^2 \frac{{g_i}^e {g_i}^f}{s-M_Z^2}$ in each of them, with $g_z = g/(2c_w)$ and $1/(s - M_Z^2)$ being the Z^0 propagator, to show:

$$|\mathcal{M}_{ii}|^2 = s^2 \left| \frac{g_z^2}{s - M_Z^2} \right|^2 ({g_i}^e)^2 ({g_i}^f)^2 (1 + \cos\theta)^2$$

$$|\mathcal{M}_{ij}|^2 = s^2 \left| \frac{g_z^2}{s - M_Z^2} \right|^2 ({g_i}^e)^2 ({g_j}^f)^2 (1 - \cos\theta)^2 \quad i \neq j$$

where the indices i, j refer to the helicity (R or L) of the electrons e and fermions f, respectively, with $g_{L,R} \equiv (c_V \pm c_A)/2$. Draw a spin diagram for each amplitude.

c) Using $\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\overline{|\mathcal{M}_Z|^2}$, integrating over $\cos\theta$, and making use of the partial widths Γ_e and Γ_f calculated in Exercise 21, arrive to the expression: $\sigma_Z = 12\pi \left(\frac{\Gamma_e \Gamma_f}{M_Z^2}\right) \frac{s}{(s-M_z^2)^2}$.

d) The resonance width Γ_Z corresponds to the imaginary part of the energy, that describes the particle *decay*, given that $\psi \propto e^{iMt} e^{-\Gamma t/2}$ implies $\psi^* \psi \propto e^{-\Gamma t} = e^{-t/\tau}$, with Γ being its total width. Complete the derivation by making the substitution $M_Z \to M_Z - i\Gamma_Z/2$ in the propagator expression, assuming that $\Gamma_Z \ll M_Z$, and then do | propagator $|^2$.

Exercise 23 ($\star\star$)

Non conservation of CP-symmetry

Let us consider a generic charged current process between quarks $ab \to cd$, governed by two elements of the V_{CKM} matrix, with amplitude (we leave out the factor $G_F/\sqrt{2}$ in what follows):

$$\mathcal{M} = V_{ca}V_{db}^* \left[\bar{u}_c\gamma^\mu(1-\gamma^5)u_a\right]\left[\bar{u}_b\gamma_\mu(1-\gamma^5)u_d\right]^+ = V_{ca}V_{db}^* \left[\bar{u}_c\gamma^\mu(1-\gamma^5)u_a\right]\left[\bar{u}_d\gamma_\mu(1-\gamma^5)u_b\right]$$

It is taken as known that the hermitic conjugate amplitude $\mathcal{M}^+$ is obtained by inverting the fermion arrows of the charged currents in the Feynman diagram (i.e. by running the time backwards). Hence:

$$\mathcal{M}^+ = V_{ca}^*V_{db}\left[\bar{u}_a\gamma^\mu(1-\gamma^5)u_c\right]\left[\bar{u}_b\gamma_\mu(1-\gamma^5)u_d\right]$$

By first applying the CP operator product on every spinor in $\mathcal{M}$, *show* that the CP-conjugate amplitude takes the form:

$$\mathcal{M}_{CP} = (J^\mu_{ca})_{CP}(J^+_{\mu,bd})_{CP} = V_{ca}V_{db}^*\left[\bar{u}_a\gamma^\mu(1-\gamma^5)u_c\right]\left[\bar{u}_b\gamma_\mu(1-\gamma^5)u_d\right]$$

Now **the necessary and sufficient condition for CP-conservation is that $\mathcal{M}_{CP} = \mathcal{M}^+$**. Thus it becomes evident that only the presence of measurable phases in the V_{CKM} matrix may cause the breakdown of CP-symmetry, when $V_{ca}V_{db}^* \neq V_{ca}^*V_{db}$.

The essential part of the proof is showing that:

$$(J^\mu_{ca})_{CP} = (-1)V_{ca}\bar{u}_a\gamma^{\mu+}(1-\gamma^5)u_c$$

For that purpose use the form these operators take in the Dirac-Pauli representation: $C = i\gamma^2\gamma^0$ and $P = \gamma^0$, together with the expression of the charge-conjugate spinors: $u_C = C\bar{u}^T$ and its adjoint $\bar{u}_C = -u^TC^{-1}$. Also take into account that $\gamma^{\mu+} = g^{0\mu}\gamma^\mu$.

Note the amplitude $\mathcal{M}_{CP}$ only differs from $\mathcal{M}^+$ (and from $\mathcal{M}$) **by one phase**, thus CP-violation can only manifest itself by means of quantum *interference* processes, with at least two amplitudes.

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