Heavy quark impurities, holography and entanglement

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Based on work with: Dorian Silvani (Swansea) (1611.06033 and 1711.01554).
Defects and impurities play a central role in various physical systems, e.g. Kondo effect.

For CFTs in 2-d with a boundary; with flow triggered on boundary (and critical bulk), the \( g \)-theorem applies: there exists a quantity \( g \) (boundary entropy) decreasing monotonically under the flow. [Affleck-Ludwig; Friedan-Konnechny]

\( g \)-function: contribution to the Entanglement Entropy (EE) of interval containing the defect.

Heavy quark “impurities” in gauge theories compute Wilson/Polyakov loops. For generic representations these are computed holographically by objects with nontrivial dynamics.

Is there a version of the \( g \)-theorem via EE for these?

What is the complete characterization of Polyakov loops at strong coupling?
Wilson loops, basic gauge-invariant Yang-Mills observables:

\[ W[C] = \frac{1}{\dim[R]} \text{Tr}_R \mathcal{P} \exp \left( i \oint_C A_\mu \dot{x}^\mu ds \right) \]

Representation “R” and contour C.

In large-\(N\) theories with holographic gravity/string duals, they are computed by (multiple) open string world sheets [Maldacena (2001); Rey-Theisen-Yee (2001)]
Wilson lines/heavy quarks

- Wilson line $\leftrightarrow$ Phase associated to heavy quark worldline.

- Heavy quark “impurity” interacts with the “ambient” Yang-Mills degrees of freedom:

$$\mathcal{L} = \mathcal{L}_{\text{imp}} + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{YM}}$$

- Probe this with:
  
  (i) $\langle \mathcal{O}_{\text{YM}} \rangle(r)$: Gauge-invariant Yang-Mills operators

  (ii) $S_{\text{EE}}(r)$: Entanglement Entropy across sphere

  (iii) Deformations of $\mathcal{L}_{\text{imp}}$
Holographic Wilson lines

- Holographic gauge/string duality for large-\(N\) theories: Natural to consider BPS Wilson lines, e.g. in \(N=4\) SUSY Yang-Mills:

\[
W[C] = \frac{1}{\dim[R]} \text{Tr}_R \mathcal{P} \exp \left[ i \oint_C (A_\mu \dot{x}^\mu + \Phi_I n^I(s)|\dot{x}|) \, ds \right]
\]

\(\{\Phi_I\}_{I=1,...,6}\): Scalars of \(\mathcal{N}=4\) theory. \(n^I\): Unit vector in \(\mathbb{R}^6\)

(Note: The Euclidean continued thermal/Polyakov line is not unitary)

- Such Wilson lines computed by 2d string worldsheets

  or

  Higher dimensional wrapped branes with world-volume

  \(\sim \mathbb{R}^2 \times M\) for some compact \(M\).
Large rank representations

- When \( \text{dim}[R] = k \) with \( \frac{k}{N} = \text{fixed} \) as \( N \to \infty \)
  
  \( k \) strings \( \rightarrow \) wrapped D-brane.

  \[ SU(N) \mathcal{N} = 4 \text{ SUSY Yang-Mills:} \]

- Rank \( k \) antisymmetric tensor \( (A_k) \): D5-brane wrapping \( S^4 \subset AdS_5 \times S^5 \)

  ![Diagram](image_url)

  [Camino-Paredes-Ramallo (2001); Hartnoll-SPK (2006); Yamaguchi (2006)]

- Rank \( k \) symmetric tensor \( (S_k) \): D3-brane wraps \( S^2 \subset AdS_5 \) [Drukker-Fiol (2005)]
Yang-Mills VEVs induced by heavy quarks

- Static quark sources Yang-Mills fields: \( \langle \text{Tr} \ F_{\mu\nu} F^{\mu\nu} \rangle, \langle T_{\mu\nu} \rangle \ldots \)

- VEVs \( \leftrightarrow \) Normalizable modes of \( AdS_5 \) fields

- \( S_{\text{bulk}} \sim N^2 S_{\text{sugra}}[\phi, g_{\mu\nu} \ldots] + \int J_{\text{source}} \)

  't Hooft coupling: \( \lambda = g_{YM}^2 N \gg 1 \)

  \( J_{F1} \sim \sqrt{\lambda} \quad J_{D3,D5} \sim N \sqrt{\lambda} \)

- Dilaton \( \phi \) \( \leftrightarrow \) Scalar “glueball” \( \sim \frac{1}{N} \text{Tr} \ F_{\mu\nu} F^{\mu\nu} \)

- Metric \( g_{\mu\nu} \) \( \leftrightarrow \) Stress tensor \( T_{\mu\nu} \)
Yang-Mills VEVs

E.g: Extracting $\langle O \rangle = \langle \frac{1}{N} \text{Tr} F_{\mu\nu} F^{\mu\nu} \rangle$:

- Solve linearized e.o.m. of dilaton in presence of source in $AdS_5$

$$N^2 \phi(\vec{x}, z) = \int d^5x' \ G_{AdS}(x'^\mu, z; x'^\mu, z') \ J(x', z')$$

[Danielsson, Kesko-Vakkuri, Kruczenski ('98)]

- Expand RHS near AdS-boundary $z \to 0$ and find coefficient of $z^4$, given $\Delta_O = 4$. 

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Conformal impurity with representation $A_k$

- The straight (BPS) Wilson line in $\mathcal{N} = 4$ SYM has $\langle W \rangle = 1$ for any representation.

- For representation $A_k$: D5-brane embedding $\simeq AdS_2 \times S^4$

\[
AdS_5 : \quad ds^2 = \frac{dz^2}{z^2} + \frac{1}{z^2} \left( -dt^2 + d\vec{x}^2 \right)
\]

\[
S^5 : \quad ds^2 = d\theta^2 + \sin^2 \theta \, d\Omega_4^2
\]

D5-brane: $(t, z, \Omega_4)$ \hspace{1cm} $\vec{x} = 0$

\[
\theta = \theta(z)
\]
D5-brane embedding

\[ S_{D5} = N \frac{\sqrt{\lambda}}{8\pi^2} \left[ \int dt \, dr \, d\Omega_4 \sqrt{\ast g} + 2\pi \alpha' F - \int 2\pi \alpha' F \wedge \ast C_4 \right] \]

\[ \frac{\delta S}{\delta F} = -k \]

- World-volume electric field fixes number of units of string charge.

- E.o.m. yields a constant solution: \( \theta(z) = \theta_k \)
  \[ \frac{\pi k}{N} = \theta_k - \sin \theta_k \cos \theta_k \]

- World-volume \( \simeq AdS_2 \times S^4 \implies \) Conformal quantum mechanics on impurity
Conformal impurity with representation $A_k$

- Lightest (linearized) fluctuation of D5-brane embedding: $\delta \theta(z)$

\[
m^2 = 12, \quad \Delta = \frac{1}{2} + \sqrt{\frac{1}{4} + m^2} = 4
\]

Irrelevant operator in CFT$_1$

- For conformal impurity $\langle \text{Tr} F^2 \rangle \sim r^{-4}$ where ($r = |\vec{x}|$):

\[
\frac{1}{N} \langle \text{Tr} F^2 \rangle_{A_k} = \frac{\sqrt{2}}{24\pi^2} \sqrt{\lambda} \sin^3 \theta_k \frac{1}{r^4}
\]

- Same factor appears as action for circular Wilson loop and Polyakov loop in rep. $A_k$
The wrapped D5-brane permits a distinct finite action solution with $\theta = \pi$.

Also conformal with $AdS_2$ factor and action of $k$ strings or $k$ fundamental quarks

Lightest fluctuation $\delta \theta(z)$: $m^2 = 0 \leftrightarrow \Delta = 1$ in $CFT_1$

$$\frac{1}{N} \langle Tr F^2 \rangle_{\text{collapsed}} = \frac{\sqrt{2}}{24\pi^2} \sqrt{\lambda} \frac{3\pi k}{2N} \frac{1}{r^4}$$

Strength: $k$ coincident strings/quarks
Interpolating solution

- Exact interpolating solution (BPS):

\[ \frac{1}{z(\theta)} = \frac{A}{\sin \theta} (\theta - \sin \theta \cos \theta - \frac{\pi k}{N})^{1/3} \]

[Callan-Guijosa-Savvidy (1998)]

- \( k \) fundamental reps. in UV (small \( z \)) \( \rightarrow \mathcal{A}_k \) in IR (large \( z \))

- For small \( z \) (UV)

\[ \theta(z) \simeq \pi - A z \ldots \]

Interpreted as a VEV for the \( \Delta = 1 \) operator in CFT\( _1 \)
Interpolating $\frac{1}{N} \langle \text{Tr} F^2 \rangle$

- Interpretation: $k$ fundamental sources screened to antisymmetric representation. [SPK-Silvani 1611.06033]
Symmetric representation $S_k$ and deformation

- **D3-brane embedding for rep. $S_k$:** $AdS_2 \times S^2$

  \[ AdS_5 : \quad ds^2 = \frac{dz^2}{z^2} + \frac{1}{z^2} (-dt^2 + d\rho^2 + \rho^2 d\Omega_2^2) \]

  
  \[ D3 : \quad (t, z, \rho, \Omega_2) \quad \rho = \kappa z \quad \kappa \equiv \frac{\sqrt{\lambda k}}{4N} \]

- "Breathing mode" of $S^2$ has $m^2 = 0$: $\Delta = 1$ in CFT$_1$

- Exact non-conformal (BPS) embedding:

  \[ \rho = \frac{\kappa z}{1 + a \kappa z} \]
Two non-conformal solutions

- $a > 0$: Rep. $S_k$, small $a z$ (UV) $\rightarrow$ $k$ fundamental quarks, large $a z$ (IR)

- $a < 0$: Rep. $S_k$ heavy quark in a Coulomb phase with $SU(N) \rightarrow U(1) \times SU(N - 1)$
Interpolating solution

- **Small** $z$: \( \frac{\rho}{z} = \kappa - a\kappa^2 z \ldots \)

Flow triggered by VEV of $\Delta = 1$ operator in UV

\[
\frac{\langle O_{F^2} \rangle_{D3}}{\langle O_{F^2} \rangle_{D0}}
\]

- **UV**: \( \frac{1}{N} \langle \text{Tr} F^2 \rangle \rightarrow \frac{\sqrt{2}}{4\pi} \kappa \sqrt{1 + \kappa^2} \rho^{-4} \)

(Fiol, Garolera, Lewkowycz (2012))

- **IR**: \( (a > 0) \rightarrow \frac{\sqrt{2}}{16\pi} \sqrt{\lambda} \frac{k}{N} \rho^{-4} \quad a < 0 : \rightarrow \frac{\sqrt{2}}{4\pi} \kappa^2 \rho^{-4} \)

- \( (a > 0) \) soln: Symmetric rep. source “dissociating” into fundamental quarks

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Heavy quark impurities, holography and entanglement
All non-conformal impurities describe flows triggered by VEV for a $\Delta = 1$ mode in CFT$_1$

All accompanied by a decrease of the strength of source towards the IR: suggesting a “thinning out” of degrees of freedom

We turn to an alternative measure: The contribution from the impurity to the Entanglement Entropy of a spherical region enclosing it
EE measures the degree of quantum entanglement in a given QFT state

For a subsystem $A$, defined via the reduced density matrix $\rho_A$

$$S_{EE}(A) = -\text{Tr} \rho_A \ln \rho_A$$

Computed using the replica trick:

$$S_{EE}(A) = - \lim_{n \to 1} \frac{\partial}{\partial n} \text{Tr} \rho_A^n$$

The holographic Ryu-Takayanagi minimal area surface prescription cannot be immediately applied for external probes. Need backreaction from the probes.
Implement replica trick in the bulk. [Lewkowycz-Maldacena (2013)]

We are interested in the EE excess due to impurity for a spherical region $\mathcal{B}$ of radius $r$ centred around the impurity.

Map the causal development $\mathcal{D}$ of the $\mathcal{B}$ to the Rindler wedge, which is conformal to hyperbolic space $H_3$ at temperature $\beta = \frac{1}{2\pi}$

The bulk dual of $\mathcal{D}$ is hyperbolically sliced AdS$_5$ with horizon and Hawking temperature $\beta$.
The excess EE due to impurity given by varying probe action wrt $\beta$:

$$S_{EE}^\text{imp}(r) = \lim_{\beta \to 2\pi} \beta \partial_\beta S_{D3,D5}(\beta)$$

(Importantly, one may use the embedding at $\beta = 2\pi$ to evaluate the action above [Karch, Uhlemann (2014)].)
Impurity EE for screening to $\mathcal{A}_k$

\[ \frac{S_{D5}}{k S^\Box} \bigg|_{A_r \to 0} = 1, \quad \frac{S_{D5}}{k S^\Box} \bigg|_{A_r \to \infty} = \frac{2N}{3\pi k} \sin^3 \theta_k. \]

- **Non-monotonic**, settling at lower value in deep IR, and qualitatively tracking $\langle \text{Tr} F^2 \rangle$
Impurity EE for flow from $S_k$

\[
\begin{align*}
S_{D3}|_{ar \rightarrow 0} &= S_{S_k} = N \left( \sinh^{-1} \kappa - \frac{1}{3} \kappa \sqrt{1 + \kappa^2} \right) \\
S_{D3}|_{ar \rightarrow \infty} &= kS_\Box = \frac{2}{3} N \kappa
\end{align*}
\]

- **Non-monotonic**, not positive definite, settling at higher IR value, disagrees qualitatively with $\langle \text{Tr} F^2 \rangle$
Coulomb Branch $S_k$

\[ S_{D3} |_{aR \gg 1} = N \left[ -\frac{1}{3} (|a| \tilde{r})^2 - |a| r + \frac{2}{3} \ln \left(2 |a| \tilde{r} \right) \right] . \]
On general grounds and specifically by considering brane-intersections impurity theory has the form (Gomis-Passerini (2006)):

\[
S_{\text{imp}} = S_{N=4} + \int dt \left[ i\bar{\chi}_m \partial_t \chi^m + \bar{\chi}_m \left( A_0 + \hat{n}^l \phi^l \right)^m_n \chi^n + \mu \left( \bar{\chi}_m \chi^m - k \right) \right].
\]

\{\chi_m\} are \(N\) fermions (bosons) for antisymmetric (symmetric) Wilson lines.

The bulk field \(A_0 + \hat{n}^l \phi^l\) has constant propagator (Erickson-Semenoff-Zarembo; Drukker-Gross) for circular Wilson line (and for Polyakov loops)

\[
D(t - t') \sim \frac{\lambda}{N}
\]
Integrate out \((A_0 + \hat{n}^l \phi^l)\) exactly:

\[
S_{\text{imp}} = \int_0^\beta d\tau \left( \bar{\chi}_m \partial_\tau \chi^m + \right)
\]

\[
\frac{1}{2} \int_0^\beta d\tau' D_\beta(\tau - \tau') \bar{\chi}_m(\tau) \chi^n(\tau) \bar{\chi}_n(\tau') \chi^m(\tau') \right).}

This theory can be solved exactly at large-\(N\), \(k\) and at large \(\lambda\) to yield the action \(\sim \sin^3 \theta_k\) for circular loops. \cite{Sachdev2010, Mueck2011}
Further Questions

- Use the impurity QM as a starting point to understand deformations and “flows”. In particular, the interpolation between the symmetric representation and $k$ fundamentals.

- All flows discussed are puzzling: originate from a “VEV” in the UV of the impurity theory. Spontaneous breaking of (conformal) symmetry in QM should not be possible.

- The impurity contribution to EE indicates the absence of a “$g$-theorem” for higher codimension impurities. It would be interesting to understand or prove this using purely holographic methods.